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NULL CONTROLLABILITY OF THE LOTKA-MCKENDRICK SYSTEM WITH SPATIAL DIFFUSION

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Abstract. We consider the infinite dimensional linear control system described by the population dynamics model of Lotka-McKendrick with spatial diffusion. Considering control functions localized with respect to the spatial variable but active for all ages, we prove that the whole population can be steered to zero in any positive time. The main novelty we bring is that, unlike the existing results in the literature, we can also control the population of ages very close to 0. Another novelty brought in is the employed methodology: as far as we know, the present work is the first one remarking that the null controllability of the considered system can be obtained by using the Lebeau-Robbiano strategy, originally developed for the null-controllability of the heat equation.

1. **Introduction.** We consider a linear controlled age-structured population model with spatial diffusion described by the following system:

$$\begin{cases} \partial_t p(t,a,x) + \partial_a p(t,a,x) + \mu(a) p(t,a,x) - \Delta p(t,a,x) \\ \qquad = \chi_\omega(x) u(t,a,x), & t>0,\ a\in(0,a_\dagger),\ x\in\Omega, \\ \dfrac{\partial p}{\partial \nu}(t,a,x) = 0, & t>0,\ a\in(0,a_\dagger),\ x\in\partial\Omega, \\ p(t,0,x) = \displaystyle\int_0^{a_\dagger} \beta(a) p(t,a,x)\,\mathrm{d}a, & t>0,\ x\in\Omega, \\ p(0,a,x) = p_0(a,x), & a\in(0,a_\dagger),\ x\in\Omega. \end{cases} \tag{1}$$

In the above equations:

- $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, denotes a smooth connected bounded domain and Δ is the laplacian with respect to the variable x;

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- $\dfrac{\partial}{\partial \nu}$ denotes the derivation operator in the direction of the unit outer normal to $\partial\Omega$. We thus have homogeneous Neumann boundary conditions, thus the considered population is isolated from the exterior of Ω;
- $p(t, a, x)$ denotes the distribution density of the population at time t, of age a at spatial position $x \in \Omega$;
- p_0 denotes the initial population distribution;
- $a_\dagger \in (0, +\infty)$ is the maximal life expectancy;
- $\beta(a)$ and $\mu(a)$ are positive functions denoting respectively the birth and death rates, which are supposed to be independent of t and x;
- $\omega \subset \Omega$ is a nonempty open susbet of Ω and χ_ω denotes the characteristic function of ω.

We make the following classical assumptions on β and μ:

(H1) $\beta \in L^\infty(0, a_\dagger)$, $\beta(a) \geqslant 0$ for almost every $a \in (0, a_\dagger)$,
(H2) $\mu \in L^1[0, a^*]$ for every $a^* \in (0, a_\dagger)$, $\mu(a) \geqslant 0$ for almost every $a \in (0, a_\dagger)$,
(H3) $\displaystyle\int_0^{a_\dagger} \mu(a)\, \mathrm{d}a = +\infty$.

We also introduce the function

$$\pi(a) := \exp\left(-\int_0^a \mu(s)\, \mathrm{d}s\right), \tag{2}$$

which is the probability of survival of an individual from age 0 to a.

Our main result is

Theorem 1.1. *With the above notations and assumptions, for every $\tau > 0$ and for every $p_0 \in L^2((0, a_\dagger) \times \Omega)$, there exists $u \in L^\infty((0, \tau); L^2((0, a_\dagger) \times \omega))$ such that the solution p of 1 satisfies*

$$p(\tau, a, x) = 0 \qquad (a \in (0, a_\dagger),\ x \in \Omega \quad \text{a.e.}). \tag{3}$$

The null-controllability of the the system modelling age-dependant population dynamics is by now well understood in the case in which diffusion is neglected (see Barbu, Ianelli and Martcheva [6] and Hegoburu, Magal, Tucsnak [10]). In the case when spatial diffusion is taken into account, namely for 1, the particular case when the control acts in the whole space (the case corresponding to $\omega = \Omega$) was investigated by S. Aniţa (see [5], p 148). The case when the control acts in a spatial subdomain ω was firstly studied by B. Ainseba [1], where the author proves the null controllability of the above system 1, except for a small interval of ages near zero. The case when the control acts in a spatial subdomain ω and also only for small age classes was investigated by B. Ainseba and S. Aniţa [2], for initial data p_0 in a neighborhood of the target $\tilde{p}$. Related approximate and exact controllability issues have also been studied in Ainseba and Langlais [4], Ainseba and Iannelli [3], Traore [22], Kavian and Traore [15].

We also mention that the controllability of related degenerated parabolic systems (namely of Kolmogorov type), not containing the renewal term, has been studied in Beauchard [7], Beauchard and Pravda-Starov [8].

The main novelty (with respect to the literature quoted above) brought in Theorem 1.1 is in our case u can be chosen such that 3 holds for every $a \in (0, a_+)$ instead of (δ, a_+), with $\delta > 0$, as it has been done, for instance, in [1]. We can thus find a control driving to zero the whole population, without excluding very young individuals. Moreover, we do not assume that the birth rate vanishes for small ages.

The remaining part of this work is organized as follows. In Section 2 we recall some basic results on the Lotka-McKendrick semigroup, with or without spatial diffusion, and we state a null controllability result associated to system 1 without spatial diffusion. Section 3 is devoted to study the null controllability of low frequencies for the solution of system 1. We prove the main result in Section 4, by using a version of the Lebeau-Robbiano strategy.

Notation. In all what follows, C will denote a generic constant, depending only of the coefficients in 1, on Ω and ω, whose value may change from line to line.

2. Some background on the Lotka-McKendrick semigroup.

In this section we recall, with no claim of originality, some existing results on the population semigroup for the linear age-structured model without and with spatial diffusion. In particular, we recall the structure of the spectrum of the semigroups generators, together with some controllability results concerning the free diffusion case.

2.1. The free diffusion case.

In this paragraph we remind some results on the diffusion free case, which is described by the so-called McKendrick-Von Foster model. With one exception, we do not give proofs and we refer, for instance, to Song et al. [20] or Inaba [12] for a detailed presentation of these issues.

The considered system is:

$$\begin{cases} \partial_t p(t,a) + \partial_a p(t,a) = -\mu(a)p(t,a), & t>0,\ a\in(0,a_\dagger), \\ p(t,0) = \displaystyle\int_0^{a_\dagger} \beta(a)p(t,a)\,\mathrm{d}a, & t>0, \\ p(0,a) = p_0(a), & a\in(0,a_\dagger), \end{cases} \tag{4}$$

where μ and β satisfy the assumptions in Theorem 1.1.

The above system is described by the operator A_0 defined by

$$\begin{aligned} \mathcal{D}(A_0) &= \left\{\varphi\in L^2[0,a_\dagger] \quad\middle|\quad \varphi(0) = \int_0^{a_\dagger}\beta(a)\varphi(a)\,\mathrm{d}a;\ -\frac{\mathrm{d}\varphi}{\mathrm{d}a} - \mu\varphi \in L^2[0,a_\dagger]\right\}, \\ A_0\varphi &= -\frac{\mathrm{d}\varphi}{\mathrm{d}a} - \mu\varphi \qquad\qquad (\varphi\in\mathcal{D}(A_0)). \end{aligned} \tag{5}$$

Theorem 2.1. *The operator A_0 defined by 5 has compact resolvent and its spectrum is constituted of a countable (infinite) set of isolated eigenvalues with finite algebraic multiplicity. The eigenvalues $(\lambda_n^0)_{n\geqslant 1}$ of A_0 (counted without multiplicity) are the solutions of the characteristic equation*

$$F(\lambda) := \int_0^{a_\dagger} \beta(a)e^{-\lambda a}\pi(a)\,\mathrm{d}a = 1. \tag{6}$$

The eigenvalues $(\lambda_n^0)_{n\geqslant 1}$ are of geometric multiplicity one, the eigenspace associated to λ_n^0 being the one-dimensional subspace of $L^2(0,a_\dagger)$ generated by the function

$$\varphi_n^0(a) = e^{-\lambda_n^0}\pi(a) = e^{-\lambda_n^0 a - \int_0^a \mu(s)\,\mathrm{d}s}.$$

Finally, every vertical strip of the complex plane $\alpha_1 \leqslant \mathrm{Re}(z) \leqslant \alpha_2$, $\alpha_1,\alpha_2\in\mathbb{R}$, contains a finite number of eigenvalues of A_0.

Theorem 2.2. *The operator A_0 defined by 5 has a unique real eigenvalue λ_1^0. Moreover, we have the following properties :*

1. *λ_1^0 is of algebraic multiplicity one;*

2. $\lambda_1^0 > 0$ *(resp.* $\lambda_1^0 < 0$*) if and only if* $F(0) > 1$ *(resp.* $F(0) < 1$*);*
3. λ_1^0 *is a real dominant eigenvalue:*

$$\lambda_1^0 > \mathrm{Re}(\lambda_n^0), \qquad \forall n \geqslant 2. \tag{7}$$

It is well known (see, for instance, Song et al. [20] or Kappel and Zhang [14]) that A generates a C^0 semigroup of linear operators in $L^2[0, a_\dagger]$ which we denote by $\mathbb{T}^{A_0} = (\mathbb{T}_t^{A_0})_{t\geqslant 0}$. We also have the following useful result (see, for instance, [12, p 23]):

Proposition 1. *The semigroup* $\mathbb{T}^{A_0}$ *generated on* $L^2([0, a_\dagger])$ *by* A_0 *is compact for* $t \geqslant a_\dagger$.

According to Zabczyk [23, Section 2]), this implies in particular that

$$\omega_a(A_0) = \omega_0(A_0),$$

where $\omega_a(A_0) := \lim\limits_{t\to+\infty} t^{-1} \ln \|\mathbb{T}_t^{A_0}\|_{L^2(0,a_\dagger)}$ denotes the growth bound of the semigroup $\mathbb{T}_t^{A_0}$ and $\omega_0(A_0) := \sup\{\mathrm{Re}\lambda \mid \lambda \in \sigma(A_0)\}$ the spectral bound of A_0. It is worth noticing that the above condition ensures that the exponential stability of $\mathbb{T}^{A_0}$ is equivalent to the condition $\omega_0(A_0) < 0$. According to Theorem 2.1 and 2.2, it follows that the exponential stability of $\mathbb{T}^{A_0}$ is equivalent to the condition $\lambda_1^0 < 0$, where $\lambda_1^0 < 0$ is the unique real solution to the characteristic equation defined by 6.

The free diffusion control problem associated to system 1 writes as

$$\begin{cases} \partial_t p(t,a) + \partial_a p(t,a) + \mu(a)p(t,a) = v(t,a), & t > 0,\ a \in (0, a_\dagger), \\ p(t,0) = \displaystyle\int_0^{a_\dagger} \beta(a)p(t,a)\,\mathrm{d}a, & t > 0, \\ p(0,a) = p_0(a), & a \in (0, a_\dagger), \end{cases} \tag{8}$$

where v is the control function. Let us state a null controllability result for system 8:

Proposition 2. *Under the assumptions of Theorem 1.1, for every* $\tau > 0$*, there exists* $v \in L^\infty((0,\tau); L^2(0,a_\dagger))$ *such that the solution* p *of 8 satisfies*

$$p(\tau, a) = 0 \qquad\qquad (a \in (0, a_\dagger) \text{ a.e.}).$$

Moreover, we have

$$\|v\|_{L^\infty((0,\tau);L^2(0,a_\dagger))} \leqslant \frac{\sup\limits_{\sigma\in[0,\tau]} \|\mathbb{T}_\sigma^{A_0}\|_{L^2(0,a_\dagger)}}{\tau} \|p_0\|_{L^2(0,a_\dagger)}. \tag{9}$$

Proof. Let $\tau > 0$. For almost every $(t,a) \in (0,\tau) \times (0, a_\dagger)$, we set

$$v(t,a) = -\frac{1}{\tau}(\mathbb{T}_t^{A_0} p_0)(a) \quad (t \in (0,\tau),\ a \in (0,a_\dagger)). \tag{10}$$

It is easy to check that the (mild) solution p of 8, given by

$$p(t,a) = \mathbb{T}_t^{A_0} p_0(a) + \int_0^t \mathbb{T}_{t-\sigma}^{A_0} v(\sigma)(a)\,\mathrm{d}\sigma, \tag{11}$$

with v defined in 10, satisfies $p(\tau, \cdot) = 0$. Moreover, the cost stated in inequality 9 follows from the definition of the control v given by 10. □

2.2. **The population dynamics with diffusion.** The existence of a semigroup on $L^2((0,a_\dagger)\times\Omega)$ describing the linear age-structured population model with diffusion coefficient and age dependent birth and death rates, with homogeneous Neumann boundary conditions has been proved in Huyer [11] (see also [9] for the case of homogeneous Dirichlet boundary conditions).

More precisely, let $X := L^2((0,a_\dagger)\times\Omega)$ and define the following unbounded operator A on X:

$$\begin{aligned}&\mathcal{D}(A)=\{\varphi\in X \quad|\quad \varphi\in C([0,a_\dagger];L^2(\Omega))\cap L^2([0,a_\dagger],H^1(\Omega)),\\ &\varphi(0,x)=\textstyle\int_0^{a_\dagger}\beta(a)\varphi(a,x)\,\mathrm{d}a;\ -\dfrac{\mathrm{d}\varphi}{\mathrm{d}a}-\mu\varphi+\Delta\varphi\in X\},\\ &A\varphi=-\dfrac{\mathrm{d}\varphi}{\mathrm{d}a}-\mu\varphi+\Delta\varphi \qquad\qquad (\varphi\in\mathcal{D}(A)).\end{aligned} \tag{12}$$

The generator A of the population semigroup can be seen as the sum of a population operator without diffusion $-\mathrm{d}/\mathrm{d}a-\mu I$ and a spatial diffusion term Δ. It turns out that spectral properties of A can be easily obtained from those of these two operators.

Theorem 2.3. *Let $0=\lambda_0<\lambda_1\leqslant\lambda_2\leqslant\dots$ be the increasing sequence of eigenvalues of $-\Delta$ with Neumann boundary conditions and let $(\varphi_n)_{n\geqslant0}$ be a corresponding orthonormal basis of $L^2(\Omega)$. Let $(\lambda_n^0)_{n\geqslant1}$ and $(\varphi_n^0)_{n\geqslant1}$ be respectively the sequence of eigenvalues and eigenfunctions of the free diffusion operator A_0 defined by 5 (see Theorem 2.1). Then the following assertions hold:*

1. *The eigenvalues of A are given by*

$$\sigma(A)=\{\lambda_i^0-\lambda_j\mid i\in\mathbb{N}^*,\ j\in\mathbb{N}\}.$$

2. *A has a dominant eigenvalue:*

$$\lambda_1=\lambda_1^0>\mathrm{Re}(\lambda),\qquad \forall\lambda\in\sigma(A),\ \lambda\neq\lambda_1.$$

3. *The eigenspace associated to an eigenvalue λ of A is given by*

$$\mathrm{Span}\{\varphi_i^0(a)\varphi_j(x)=e^{-\lambda_i^0a}\pi(a)\varphi_j(x)\mid\lambda_i^0-\lambda_j=\lambda\}.$$

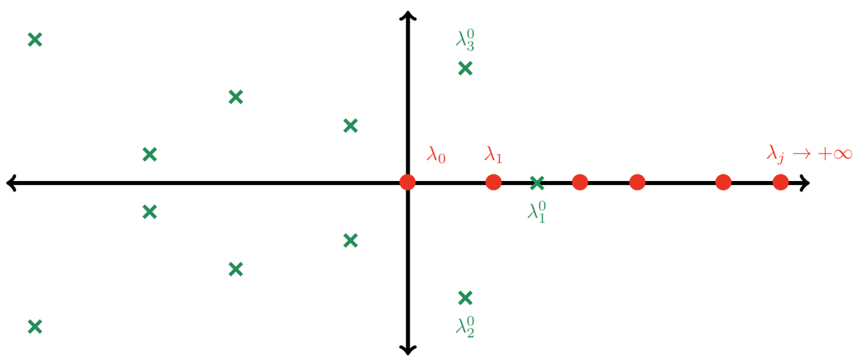

FIGURE 1. The spectrum of the free diffusion operator A_0 (green crosses) and of $-\Delta$ (red circles)

Since the operator A generates a $\mathcal{C}^0$ semigroup of linear operators in X which we denote by $\mathbb{T}^A = (\mathbb{T}^A_t)_{t\geqslant 0}$, this allows to define the concept of (mild) solution of 1 in the following standard way: we say that p is a mild solution of 1 if

$$p(t,\cdot) = \mathbb{T}^A_t p_0 + \Phi_t u \qquad (t \geqslant 0, \quad u \in L^2([0,\infty);X)), \tag{13}$$

where the control operator $B \in \mathcal{L}(X)$ is defined by

$$Bu = \chi_\omega u \qquad (u \in X),$$

and where

$$\Phi_t u = \int_0^t \mathbb{T}^A_{t-\sigma} Bu(\sigma)\,\mathrm{d}\sigma \qquad (t \geqslant 0, \quad u \in L^2([0,\infty);X)). \tag{14}$$

It is worth noticing, for instance by using a spectral decomposition, that the semigroup $\mathbb{T}^A$ is exponentially stable if $\lambda_1^0 < 0$, where we recall that λ_1^0 denotes the unique real solution to the characteristic equation defined by 6.

Remark 1. In order to prove the null controllability of system 1, we may assume, without loss of generality, that the so called reproductive number satisfies

$$\int_0^{a_\dagger} \beta(a)\pi(a)\,\mathrm{d}a < 1, \tag{15}$$

which implies that both semigroups $\mathbb{T}^{A_0}$ and $\mathbb{T}^A$ are exponentially stable - see the above results. Indeed, in the case where $\int_0^{a_\dagger} \beta(a)\pi(a)\,\mathrm{d}a \geqslant 1$, we may consider the auxiliary system

$$\begin{cases} \partial_t z(t,a,x) + \partial_a z(t,a,x) + \tilde{\mu}(a) z(t,a,x) - \Delta z(t,a,x) \\ \qquad = \chi_\omega(x) v(t,a,x), & t>0,\ a\in(0,a_\dagger),\ x\in\Omega, \\ \dfrac{\partial z}{\partial \nu}(t,a,x) = 0, & t>0,\ a\in(0,a_\dagger),\ x\in\partial\Omega, \\ z(t,0,x) = \displaystyle\int_0^{a_\dagger} \beta(a) z(t,a,x)\,\mathrm{d}a, & t>0,\ x\in\Omega, \\ z(0,a,x) = p_0(a,x), & a\in(0,a_\dagger),\ x\in\Omega, \end{cases} \tag{16}$$

with $\tilde{\mu}(a) := \mu(a) + \lambda$, where $\lambda \geqslant 0$ is large enough to have

$$\int_0^{a_\dagger} \beta(a) e^{-\int_0^a \tilde{\mu}(s)\,\mathrm{d}s}\,\mathrm{d}a < 1.$$

Suppose that the above system 16 is null controllable with control function v. Then, system 1 is null controllable with control function $u = e^{\lambda t} v$, which has the same regularity as v.

From now on, without loss of generality (see Remark 1), we assume that both semigroups $\mathbb{T}^{A_0}$ and $\mathbb{T}^A$ are exponentially stable, which implies that there exists a constant $C \geqslant 0$ and a constant $M \geqslant 0$ such that for every $t \geqslant 0$, we have

$$\|\mathbb{T}^{A_0}_t\|_{\mathcal{L}(L^2[0,a_\dagger])} \leqslant C \quad \text{and} \quad \|\mathbb{T}^A_t\|_{\mathcal{L}(X)} \leqslant M \quad (t \geqslant 0). \tag{17}$$

3. **Low frequency control.** In this section, we prove that the projection of the state trajectory of 1 on an infinite subspace of X (defined using the eigenfunctions of the Neumann Laplacian) can be steered to zero in any time and we estimate the norm of the associated control. More precisely, let $\{\varphi_j\}_{j\geqslant 0}$ be an orthonormal basis in $L^2(\Omega)$ formed of eigenvectors of the Neumann Laplacian and let $(\lambda_j)_{j\geqslant 0}$ be the

corresponding non decreasing sequence of eigenvalues. In other words $(\varphi_j)_{j\geqslant 0}$ is an orthonormal basis in $L^2(\Omega)$ such that for every $j \geqslant 0$ we have

$$\begin{cases} -\Delta\varphi_j = \lambda_j \varphi_j & \text{in } \Omega, \\ \dfrac{\partial \varphi_j}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{18}$$

In the sequel, for any $\mu > 0$, we denote by

$$\mathcal{N}(\mu) := \mathrm{Card}\{k : \lambda_k \leqslant \mu\}, \tag{19}$$

$$E_\mu := \mathrm{Span}\{\varphi_k \ : \lambda_k \leqslant \mu\},$$

and $\Pi_{E_\mu} : L^2(\Omega) \to L^2(\Omega)$ the orthogonal projection from $L^2(\Omega)$ onto E_μ. The main result of this section is:

Proposition 3. *Let $\mu > 0$, and let $T > 0$. There exists $u_\mu \in L^\infty((0,T);X)$ such that the solution p of 1 satisfies*

$$\Pi_{E_\mu} p(T, a, \cdot) = 0 \qquad (a \in (0, a_\dagger) \text{ a.e.})$$

Moreover, we have the following estimate:

$$\|u_\mu\|_{L^\infty((0,T);X)} \leqslant \frac{Ce^{C\sqrt{\mu}}}{T} \|p_0\|_X.$$

The main ingredient of the proof is an inequality involving the eigenfunctions of the Neumann Laplacian. This inequality, obtained in Jerison and Lebeau [13], can also obtained by combining results and methods from Lebeau and Robbiano [16], [17].

Theorem 3.1. *For any non-empty open subset ω of Ω, there exists $C > 0$ such that for any $\mu > 0$, for any sequence $(a_j)_{j\geqslant 0} \subset \mathbb{R}$, we have*

$$\sum_{j:\lambda_j\leqslant\mu} |a_j|^2 \leqslant Ce^{C\sqrt{\mu}} \int_\omega \left| \sum_{j:\lambda_j\leqslant\mu} a_j\varphi_j(x) \right|^2 \mathrm{d}x. \tag{20}$$

We will also use the following classical lemma, whose proof can be found, for instance, in [21] - Section 2.

Lemma 3.2. *Suppose that W, Y, Z are Hilbert spaces, $\mathcal{F} \in \mathcal{L}(W,Z)$ and that $\mathcal{G} \in \mathcal{L}(Y,Z)$. Denote by A^* the adjoint of an operator A. Then the following statements are equivalent:*

(i) *there exists $c > 0$ such that*

$$\|\mathcal{F}^* z\|_W \leqslant c\|\mathcal{G}^* z\|_Y \quad (z \in Z);$$

(ii) *there exists $\mathcal{H} \in \mathcal{L}(W,Y)$ such that $\mathcal{G}\mathcal{H} = \mathcal{F}$ and $\|\mathcal{H}\|_{\mathcal{L}(W,Y)} \leqslant c$.*

Proof of Proposition 3. Note that the solution p of 1 writes

$$p(t,a,\cdot) = \sum_{j=0}^{+\infty} p^j(t,a)\varphi_j \ \text{ in } L^2(\Omega), \ \text{ a.e. in } (0,T)\times(0,a_\dagger),$$

where p^j denotes the solution of

$$\begin{cases} \partial_t p^j(t,a) + \partial_a p^j(t,a) + (\mu(a)+\lambda_j)p^j(t,a) = \displaystyle\int_\omega u(t,a,x)\varphi_j(x)\,\mathrm{d}x, \\ p^j(t,0) = \displaystyle\int_0^{a_\dagger} \beta(a)p^j(t,a)\,\mathrm{d}a, \\ p^j(0,a) = p_0^j(a), \end{cases} \tag{21}$$

and where

$$p_0(a,\cdot) = \sum_{j=0}^{+\infty} p_0^j(a)\varphi_j \quad \text{in } L^2(\Omega), \text{ a.e. } a \in (0,a_\dagger).$$

The aim is to solve the following moment problem: find $u \in L^\infty((0,T);X)$ such that for every $j \in [0,\mathcal{N}(\mu)]$, we have

$$\int_\omega u_\mu(t,a,x)\varphi_j(x)\,\mathrm{d}x = v_j(t,a) \qquad ((t,a) \in (0,T)\times(0,a_\dagger) \ \text{ a.e.}), \tag{22}$$

where v_j denotes a null control associated to the system

$$\begin{cases} \partial_t p^j(t,a) + \partial_a p^j(t,a) + (\mu(a)+\lambda_j)p^j(t,a) = v_j(t,a), & t>0,\ a\in(0,a_\dagger), \\ p^j(t,0) = \displaystyle\int_0^{a_\dagger} \beta(a)p^j(t,a)\,\mathrm{d}a, & t>0, \\ p^j(0,a) = p_0^j(a), & a \in (0,a_\dagger). \end{cases} \tag{23}$$

Recall from Proposition 2 and assumptions stated in Remark 1 that for every $j \geqslant 0$, there exists $v_j \in L^\infty((0,T);L^2(0,a_\dagger))$ so that the solution p^j of the above system 23 satisfies $p^j(T,a) = 0$ for almost every $a \in (0,a_\dagger)$, with $\|v_j\|_{L^\infty((0,T);L^2(0,a_\dagger))} \leqslant \frac{C}{T}\|p_0^j\|_{L^2(0,a_\dagger)}$, where C is a constant independent of λ_j (since we can choose $v_j(t,a) = -\frac{e^{-t\lambda_j}}{T}(\mathbb{T}_t^{A_0}p_0^j)(a)$).

Let $\mu > 0$. Define the map $\mathcal{G} : L^2(\omega) \to \mathbb{R}^{\mathcal{N}(\mu)+1}$ by

$$\mathcal{G}u := \left(\int_\omega u(x)\varphi_j(x)\,\mathrm{d}x\right)_{0\leqslant j\leqslant \mathcal{N}(\mu)}.$$

It is easy to check that for every $w = (w_j)_{0\leqslant j\leqslant\mathcal{N}(\mu)} \in \mathbb{R}^{\mathcal{N}(\mu)+1}$, we have

$$\mathcal{G}^\star w = \sum_{j=0}^{\mathcal{N}(\mu)} w_j\varphi_j. \tag{24}$$

Inequality 20 ensures that for every $w \in \mathbb{R}^{\mathcal{N}(\mu)+1}$, we have

$$\|w\|_{\mathbb{R}^{\mathcal{N}(\mu)+1}} \leqslant Ce^{C\sqrt{\mu}}\|\mathcal{G}^\star w\|_{L^2(\omega)}. \tag{25}$$

By Lemma 3.2 with $W = Z = \mathbb{R}^{\mathcal{N}(\mu)+1}$, $Y = L^2(\omega)$ and $\mathcal{F} = \mathrm{Id}_{\mathcal{L}(\mathbb{R}^{\mathcal{N}(\mu)+1})}$, it follows that there exists $\mathcal{H} \in \mathcal{L}(\mathbb{R}^{\mathcal{N}(\mu)+1}, L^2(\omega))$ such that $\mathcal{G}\mathcal{H} = \mathrm{Id}_{\mathcal{L}(\mathbb{R}^{\mathcal{N}(\mu)+1})}$. Moreover, $\|\mathcal{H}\|_{\mathcal{L}(\mathbb{R}^{\mathcal{N}(\mu)+1})} \leqslant Ce^{C\sqrt{\mu}}$ so that for every $w \in \mathbb{R}^{\mathcal{N}(\mu)+1}$, there exists $u := \mathcal{H}(w) \in L^2(\omega)$ such that $\mathcal{G}u = w$ and $\|u\|_{L^2(\omega)} \leqslant Ce^{C\sqrt{\mu}}\|w\|_{\mathbb{R}^{\mathcal{N}(\mu)+1}}$.

Let $(t,a) \in (0,T)\times(0,a_\dagger)$. Setting $w := (v_j(t,a))_{0\leqslant j\leqslant\mathcal{N}(\mu)}$ where v_j is the null control defined by Proposition 2 (with $\mu(a)$ replaced by $(\mu(a)+\lambda_j)$), it follows that there exists $u_\mu(t,a) \in L^2(\omega)$ such that $Gu_\mu(t,a) = w$, *i.e.*

$$\int_\omega u_\mu(t,a,x)\varphi_j(x)\,\mathrm{d}x = v_j(t,a) \qquad (j \in [|0,\mathcal{N}(\mu)|],\ (t,a) \in (0,T)\times(0,a_\dagger) \ \text{ a.e.}), \tag{26}$$

with

$$\|u_\mu(t,a,\cdot)\|_{L^2(\omega)} \leqslant Ce^{C\sqrt{\mu}}\|w\|_{\mathbb{R}^{\mathcal{N}(\mu)+1}}. \tag{27}$$

From the above inequality 26, it follows that

$$\|u_\mu\|^2_{L^\infty((0,T);X)} \leqslant Ce^{C\sqrt{\mu}} \sum_{j=0}^{\mathcal{N}(\mu)} \|v_j\|^2_{L^\infty((0,T);L^2(0,a_\dagger))}, \tag{28}$$

where $\|v_j\|^2_{L^\infty((0,T);L^2(0,a_\dagger))} \leqslant \frac{C}{T^2}\|p_0^j\|^2_{L^2(0,a_\dagger)}$ by Proposition 2, so that from the above inequality we have

$$\|u_\mu\|^2_{L^\infty((0,T);X)} \leqslant \frac{Ce^{C\sqrt{\mu}}}{T^2}(\mathcal{N}(\mu)+1)\|p_0\|^2_X.$$

By Weyl's formula (see, for instance, Netrusov and Safarov [18] for a reminder), there exists a constant $K > 0$ such that $\mathcal{N}(\mu) \leqslant K\mu^{\frac{n}{2}}$, so that from the above formula we have

$$\|u_\mu\|_{L^\infty((0,T);X)} \leqslant \frac{Ce^{C\sqrt{\mu}}}{T}\|p_0\|_X,$$

for some constant $C \geqslant 0$. □

4. **Proof of the main result.**

In this section we prove Theorem 1.1 by applying a version of the Lebeau-Robbiano strategy, initially proposed for the heat equation. A distinctive feature of the version of this methodology we propose here is that the projected systems are infinite dimensional, being described by equations similar to the Lotka-McKendrick system without spatial diffusion. To apply this strategy, we need these systems to be null controllable in arbitrarily small time so that our method is limited to the case of a control which is active for all ages.

Recall from Proposition 3 that, given a time $T > 0$ and a frequency $\mu > 0$, there exists $u_\mu \in L^\infty((0,T);X)$ such that the solution p of 1 belongs to the orthogonal of E_μ at time T, for every $a \in (0,a_\dagger)$. The control cost behaves like $e^{C\sqrt{\mu}}$, and can be compensated by the natural dissipation of the solution stated in the following proposition.

Proposition 4. *Let $\mu > 0$. Suppose that $\Pi_{E_\mu} p_0(a,\cdot) = 0$ for almost every $a \in (0,a_\dagger)$. Then there exists a constant $M \geqslant 0$ such that for every $t > 0$, the solution p of 1 with $u \equiv 0$ satisfies*

$$\|p(t,\cdot,\cdot)\|_X \leqslant Me^{-\mu t}\|p_0\|_X \qquad (t \geqslant 0).$$

Proof. Suppose that $\Pi_{E_\mu} p_0(a) = 0$ for almost every $a \in (0,a_\dagger)$. With $u \equiv 0$, the solution p of 1 satisfies

$$p(t,a,\cdot) = \sum_{j:\ \lambda_j > \mu} p^j(t,a)\varphi_j \ \text{ in } L^2(\Omega), \text{ a.e. in } (0,\tau)\times(0,a_\dagger),$$

where

$$\begin{cases} \partial_t p^j(t,a) + \partial_a p^j(t,a) + (\mu(a)+\lambda_j)p^j(t,a) = 0 & t>0,\ a\in(0,a_\dagger), \\ p^j(t,0) = \displaystyle\int_0^{a_\dagger} \beta(a)p^j(t,a)\,\mathrm{d}a, & t>0, \\ p^j(0,a) = p_0^j(a), & a\in(0,a_\dagger). \end{cases} \tag{29}$$

Let $\lambda_j > \mu$. It is easy to check that the solution p^j of 29 satisfies $p_j(t,a) = e^{-t\lambda_j}\mathbb{T}_t^{A_0}(p_0^j)(a)$, so that using 17 we have

$$\begin{aligned} \|p^j(t,\cdot)\|_{L^2(0,a_\dagger)} &\leqslant Me^{-\lambda_j t}\|p_0^j\|_{L^2(0,a_\dagger)} \\ &\leqslant Me^{-\mu t}\|p_0^j\|_{L^2(0,a_\dagger)}, \end{aligned} \tag{30}$$

since $\lambda_j > \mu$. Using

$$\|p(t,a,\cdot)\|^2_{L^2(\Omega)} = \sum_{j:\ \lambda_j>\mu} |p^j(t,a)|^2,$$

it follows that we have

$$\begin{aligned} \|p(t,\cdot,\cdot)\|^2_{L^2((0,a_\dagger)\times\Omega)} &= \sum_{j:\ \lambda_j>\mu} \|p^j(t,\cdot)\|^2_{L^2(0,a_\dagger)} \\ &\leqslant M^2e^{-2\mu t}\sum_{j:\ \lambda_j>\mu}\|p_0^j\|^2_{L^2(0,a_\dagger)} \\ &\leqslant M^2e^{-2\mu t}\|p_0\|_X^2, \end{aligned} \tag{31}$$

so that the estimation of Proposition 4 holds. □

The following corollary is the key point of the strategy of Lebeau and Robbiano, originally developed for the null controllability of the heat equation [16]. Given a time $T > 0$, we construct a control in two steps in the time interval $(0,T)$: we control the first frequencies in the time interval $(0,T/2)$ with the cost obtained in Proposition 3, and we let the system to evolve freely in the time interval $(T/2,T)$ in order to dissipate the energy possibly transferred to the high frequencies (see Proposition 4).

Given Proposition 3 and Proposition 4, the following Corollary 1 and the proof of Theorem 1.1 can be obtained as a consequence of Theorem 2.1 from [8]. For the sake of completeness, we provide detailed proofs.

Corollary 1. *For every $\mu > 0$ and every $T\in(0,\tau)$, there exists $u_\mu \in L^\infty((0,T);X)$ such that*

$$\|u_\mu\|_{L^\infty((0,T);X)} \leqslant \frac{Ce^{C\sqrt{\mu}}}{T}\|p_0\|_X,$$

and

$$\|p(T,\cdot,\cdot)\|_X \leqslant Ce^{C\sqrt{\mu}-\frac{T\mu}{2}}\|p_0\|_X, \tag{32}$$

where p is the solution of 1 with control function u_μ.

Proof. Let $T\in(0,\tau)$. We first use Proposition 3 on the time interval $(0,T/2)$. This gives us a control function $v_\mu \in L^\infty((0,T/2);X)$ such that the solution p of 1 with control function v_μ satisfies

$$\Pi_{E_\mu} p(T/2,a) = 0 \qquad (a\in(0,a_\dagger)\ \text{a.e.}), \tag{33}$$

and

$$\|v_\mu\|_{L^\infty((0,T/2);X)} \leqslant \frac{Ce^{C\sqrt{\mu}}}{T}\|p_0\|_X. \tag{34}$$

In the second half interval $(T/2, T)$, we choose a null control in order to take advantage of the natural dissipation given by Proposition 4. More precisely, for almost every $t \in (0, T)$, we set

$$u_\mu(t) = \begin{cases} v_\mu(t) & \text{if } t \in (0, T/2), \\ 0 & \text{if } t \in (T/2, T). \end{cases} \tag{35}$$

Thanks to 34 and 35, it is clear that we have

$$\|u_\mu\|_{L^\infty((0,T);X)} \leqslant \frac{Ce^{C\sqrt{\mu}}}{T} \|p_0\|_X, \tag{36}$$

and since $u_\mu = 0$ on the time interval $(T/2, T)$, formula 13 gives

$$p(T) = \mathbb{T}^A_{T/2} p(T/2), \tag{37}$$

so that using Proposition 4 together with 33 and 37, it follows that we have

$$\|p(T, \cdot, \cdot)\|_X \leqslant M e^{-\frac{T\mu}{2}} \|p(T/2, \cdot, \cdot)\|_X. \tag{38}$$

Moreover, it follows from formula 13 and 35 that we have

$$p(T/2) = \mathbb{T}^A_{T/2} p_0 + \int_0^{T/2} \mathbb{T}^A_{T/2-\sigma} B v_\mu(\sigma)\, \mathrm{d}\sigma,$$

and it follows from the assumption 17 and Cauchy-Schwarz inequality that we have

$$\begin{aligned} \|p(T/2, \cdot, \cdot)\|_X &\leqslant M\|p_0\|_X + M\sqrt{T/2}\|v_\mu\|_{L^2((0,T/2);X)} \\ &\leqslant M\|p_0\|_X + \frac{M}{2} T \|v_\mu\|_{L^\infty((0,T/2);X)} \end{aligned} \tag{39}$$

so that using 34, 38 and 39 we have

$$\|p(T, \cdot, \cdot)\|_X \leqslant Ce^{C\sqrt{\mu} - \frac{T\mu}{2}} \|p_0\|_X, \tag{40}$$

for some constant $C \geqslant 0$. □

We then use a time-splitting procedure, as described in [16], to get the null controllability of system 1.

Proof of theorem 1.1. Let $\tau > 0$. Consider the sequences

$$T_j = \frac{\tau}{2^j} \quad \text{and} \quad \mu_j = \beta(2^j)^2 \quad (j \geqslant 1),$$

where $\beta > 0$ is determined later. Denote by $\tau_0 = 0$ and $\tau_j = \sum_{k=1}^{j} T_k$, for every $j \geqslant 1$, so that we have $(0, \tau) = \bigcup_{j \geqslant 0} (\tau_j, \tau_{j+1})$. Following the strategy of Lebeau and Robbiano [16], we will define a control by induction on each interval (τ_j, τ_{j+1}) which drives the initial state to zero in time τ.

Firstly, during the time interval $(0, \tau_1) = (0, T_1)$, we apply a control u_1 as given by Corollary 1 with $\mu = \mu_1$, so that we have

$$\|u_1\|_{L^\infty((0,\tau_1);X)} \leqslant \frac{Ce^{C\sqrt{\mu_1}}}{T_1} \|p_0\|_X,$$

and

$$\|p(\tau_1, \cdot, \cdot)\|_X \leqslant Ce^{C\sqrt{\mu_1} - \frac{T_1\mu_1}{2}} \|p_0\|_X.$$

Given $j \geqslant 1$, during the time interval (τ_{j-1}, τ_j), we apply by induction a control function denoted by u_j as given by Corollary 1 with $\mu = \mu_j$ so that we have

$$\|u_j\|_{L^\infty((\tau_{j-1},\tau_j);X)} \leqslant \frac{Ce^{C\sqrt{\mu_j}}}{T_j}\|p(\tau_{j-1},\cdot,\cdot)\|_X, \tag{41}$$

and

$$\|p(\tau_j,\cdot,\cdot)\|_X \leqslant Ce^{C\sqrt{\mu_j}-\frac{T_j\mu_j}{2}}\|p(\tau_{j-1},\cdot,\cdot)\|_X. \tag{42}$$

From 42, it follows that for every $j \geqslant 1$ we have

$$\|p(T_j,\cdot,\cdot)\|_X \leqslant C^j e^{C\sum_{k=1}^j\sqrt{\mu_k}-\frac{1}{2}\sum_{k=1}^j T_k\mu_k}\|p_0\|_X, \tag{43}$$

with

$$C\sum_{k=1}^j\sqrt{\mu_k} - \frac{1}{2}\sum_{k=1}^j T_k\mu_k = 2(2^j-1)(C\sqrt{\beta}-\frac{\beta}{2}\tau), \tag{44}$$

by construction of the sequences $(T_j)_{j\geqslant 1}$ and $(\mu_j)_{j\geqslant 1}$. Then we choose $\beta > 0$ large enough so that

$$\tilde{\beta} := \frac{\beta}{2}\tau - C\sqrt{\beta} > 0, \tag{45}$$

so that from 43, 44 and 45 we have

$$\|p(\tau_j,\cdot,\cdot)\|_X \leqslant KC^j e^{-\tilde{\beta}2^{j+1}}\|p_0\|_X, \tag{46}$$

for some constant $K \geqslant 0$.

Going back to 41 and using the estimate given by 46, it follows that we have

$$\begin{aligned}\|u_j\|_{L^\infty((\tau_{j-1},\tau_j);X)} &\leqslant \frac{KC}{\tau}e^{C\sqrt{\beta}2^j}2^{-j}C^{j-1}e^{-2\tilde{\beta}(2^{j-1}-1)}\|p_0\|_X \\ &= \frac{\tilde{K}}{\tau}C^j 2^{-j}e^{2^j(C\sqrt{\beta}-\tilde{\beta})}\|p_0\|_X,\end{aligned} \tag{47}$$

with $\tilde{K} = Ke^{2\tilde{\beta}}$. We choose $\beta > 0$ large enough such that

$$\bar{\beta} := \tilde{\beta} - C\sqrt{\beta} = \frac{\beta}{2}\tau - 2C\sqrt{\beta} > 0, \tag{48}$$

so that from 47 and 48 we have

$$\|u_j\|_{L^\infty((\tau_{j-1},\tau_j);X)} \leqslant \frac{\tilde{K}}{\tau}C^j 2^{-j}e^{-\bar{\beta}2^j}\|p_0\|_X. \tag{49}$$

The above estimate 49 ensures that

$$\sup_{j\geqslant 1}\|u_j\|_{L^\infty((\tau_{j-1},\tau_j);X)} < +\infty, \tag{50}$$

so that defining the control u by

$$u = \sum_{j=1}^{+\infty} u_j \mathbb{1}_{(\tau_{j-1},\tau_j)} \tag{51}$$

gives a control function in $L^\infty((0,\tau);X)$. The corresponding trajectory p is continuous in time with values in X and satisfies

$$\|p(\tau_j,\cdot,\cdot)\|_X \underset{j\to\infty}{\longrightarrow} 0, \tag{52}$$

thanks to estimation 46. This implies that $p(\tau,\cdot,\cdot) = 0$, since $\tau_j \to \tau$ as $j \to +\infty$. □

Remark 2. Given $\tau > 0$, in the above proof, we can choose $\beta = 2(\frac{4C}{\tau})^2$ so that the condition 48 : $\bar{\beta} := \frac{\beta}{2}\tau - 2C\sqrt{\beta} > 0$ is fulfilled (in this case, we have $\bar{\beta} = \frac{4c^2}{\tau}(4 - 2\sqrt{2})$). With this choise of β, it follows from 49 that the control u defined by 51 satisfies

$$\|u\|_{L^\infty((0,\tau);X)} \leqslant K e^{\frac{K}{\tau}} \|p_0\|_X, \tag{53}$$

for some constant $K \geqslant 0$. The same type of estimation is shown by Seidman [19] for the null controllability of the heat equation, using also an adaptated Lebeau-Robbiano strategy.

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