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TIME OPTIMAL INTERNAL CONTROLS FOR THE LOTKA-MCKENDRICK EQUATION WITH SPATIAL DIFFUSION

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Abstract. This work is devoted to establish a bang-bang principle of time optimal controls for a controlled age-structured population evolving in a bounded domain of $\mathbb{R}^n$. Here, the bang-bang principle is deduced by an L^∞ null-controllability result for the Lotka-McKendrick equation with spatial diffusion. This L^∞ null-controllability result is obtained by combining a methodology employed by Hegoburu and Tucsnak - originally devoted to study the null-controllability of the Lotka-McKendrick equation with spatial diffusion in the more classical L^2 setting - with a strategy developed by Wang, originally intended to study the time optimal internal controls for the heat equation.

1. Introduction.

We consider a linear controlled age-structured population model with spatial diffusion described by the following system:

$$\begin{cases} \partial_t p(t,a,x) + \partial_a p(t,a,x) + \mu(a)p(t,a,x) - \Delta p(t,a,x) & \\ \quad = \chi_\omega(x)u(t,a,x), & t>0,\ a\in(0,a_\dagger),\ x\in\Omega, \\ \dfrac{\partial p}{\partial \nu}(t,a,x) = 0, & t>0,\ a\in(0,a_\dagger),\ x\in\partial\Omega, \\ p(t,0,x) = \displaystyle\int_0^{a_\dagger} \beta(a)p(t,a,x)\,\mathrm{d}a, & t>0,\ x\in\Omega, \\ p(0,a,x) = p_0(a,x), & a\in(0,a_\dagger),\ x\in\Omega. \end{cases} \tag{1}$$

In the above equations:

- $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, denotes a smooth connected bounded domain and Δ is the laplacian with respect to the variable x;
- $\dfrac{\partial}{\partial \nu}$ denotes the derivation operator in the direction of the unit outer normal to $\partial\Omega$. We thus have homogeneous Neumann boundary conditions, thus the considered population is isolated from the exterior of Ω;
- $p(t,a,x)$ denotes the distribution density of the population at time t, of age a at spatial position $x \in \Omega$;
- p_0 denotes the initial population distribution;
- $a_\dagger \in (0,+\infty)$ is the maximal life expectancy;
- $\beta(a)$ and $\mu(a)$ are positive functions denoting respectively the birth and death rates, which are supposed to be independent of t and x;

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- $\omega \subset \Omega$ is a nonempty open susbet of Ω and χ_ω denotes the characteristic function of ω.

We make the following classical assumptions on β and μ:

(H1) $\beta \in L^\infty(0, a_\dagger)$, $\beta(a) \geqslant 0$ for almost every $a \in (0, a_\dagger)$,
(H2) $\mu \in L^1[0, a^*]$ for every $a^* \in (0, a_\dagger)$, $\mu(a) \geqslant 0$ for almost every $a \in (0, a_\dagger)$,
(H3) $\displaystyle\int_0^{a_\dagger} \mu(a)\,\mathrm{d}a = +\infty$.

In a recent work, Hegoburu and Tucsnak [16] proved that the above system (1) is null controllable in any time $\tau > 0$, in the sense that for any $p_0 \in L^2((0, a_\dagger) \times \Omega)$, there exists a control function $u \in L^\infty((0, \tau); L^2((0, a_\dagger) \times \omega))$ such that the corresponding solution p of (1) satisfies

$$p(\tau, a, x) = 0 \qquad ((a, x) \in (0, a_\dagger) \times \Omega \ \text{ a.e.}).$$

Our aim is to study the associated time optimal control problem, in an L^∞ setting. More precisely, given $M > 0$, we define the set of admissible controls by

$$\mathcal{U}_{ad} := \{u \in L^\infty([0, \infty) \times (0, a_\dagger) \times \omega) \ \mid \ |u(t, a, x)| \leqslant M \text{ a.e. in } [0, \infty) \times (0, a_\dagger) \times \omega\}.$$

Given $p_0 \in L^\infty((0, a_\dagger); L^2(\Omega))$, we define the set of reachable states from p_0 as

$$\mathcal{R}(p_0, \mathcal{U}_{ad}) := \{p(\tau, \cdot, \cdot) \ \mid \ \tau > 0 \text{ and } p \text{ is the solution of (1) with } u \in \mathcal{U}_{ad}\}.$$

For $p_0 \in L^\infty((0, a_\dagger); L^2(\Omega))$ and $p_1 \in \mathcal{R}(p_0, \mathcal{U}_{ad})$, the *time optimal control problem for system* (1) consists in determining an input $u^* \in \mathcal{U}_{ad}$ such that the corresponding solution p^* of (1) satisfies

$$p^*(\tau^*(p_0, p_1)) = p_1,$$

where $\tau^*(p_0, p_1)$ is the minimal time needed to steer the initial data p_0 towards the target population p_1 with controls in $\mathcal{U}_{ad}$,

$$\tau^*(p_0, p_1) = \min_{u \in \mathcal{U}_{ad}} \{\tau \ \mid \ p(\tau, \cdot, \cdot) = p_1\}. \tag{2}$$

The main result in this work asserts that the solution is bang-bang and unique. More precisely, we have

Theorem 1.1. *With the above notations and assumptions, for any $p_0 \in L^\infty((0, a_\dagger); L^2(\Omega))$ and any $p_1 \in \mathcal{R}(p_0, \mathcal{U}_{ad})$, there exists a unique solution u^* of the time optimal control problem* (2). *This solution u^* has the bang-bang property:*

$$|u(t, a, x)| = M \quad \text{a.e. in } [0, \infty) \times (0, a_\dagger) \times \omega. \tag{3}$$

It is know (see, for instance, Micu, Roventa and Tucsnak [32] - Proposition 2.6.) that the existence and uniqueness of the time optimal control problem (2) which has the bang-bang property (3) may be induced by the L^∞ null-controllability of the system (1) in any arbitrary time $\tau > 0$ over any subset E of positive measure in $[0, \tau]$. Hence, in order to prove the above Theorem 1.1, we need to derive from [16] the following L^∞ null controllability result for the Lotka-McKendrick equation:

Theorem 1.2. *With the above notations and assumptions, let τ be a positive constant and let E be a subset of the interval $[0, \tau]$ with positive measure. Then the system* (1) *is L^∞ null-controllable in time $\tau > 0$ over E, in the sense that for every*

$p_0 \in L^\infty((0, a_\dagger); L^2(\Omega))$, there exists a control $u \in L^\infty((0, \tau) \times (0, a_\dagger) \times \omega)$ such that the solution p of the following controlled population equation:

$$\begin{cases} \partial_t p(t, a, x) + \partial_a p(t, a, x) + \mu(a)p(t, a, x) & \\ \quad - \Delta p(t, a, x) = \chi_E(t)\chi_\omega(x)u(t, a, x), & t \in (0, \tau),\ a \in (0, a_\dagger),\ x \in \Omega, \\ \dfrac{\partial p}{\partial \nu}(t, a, x) = 0, & t \in (0, \tau),\ a \in (0, a_\dagger),\ x \in \partial\Omega, \\ p(t, 0, x) = \displaystyle\int_0^{a_\dagger} \beta(a)p(t, a, x)\, \mathrm{d}a, & t \in (0, \tau),\ x \in \Omega, \\ p(0, a, x) = p_0(a, x), & a \in (0, a_\dagger),\ x \in \Omega, \end{cases} \tag{4}$$

satisfies $p(\tau, a, x) = 0$ for almost every $(a, x) \in (0, a_\dagger) \times \Omega$. Moreover, we have

$$\|u\|_{L^\infty((0,\tau)\times(0,a_\dagger)\times\omega)} \leqslant L\|p_0\|_{L^\infty((0,a_\dagger);L^2(\Omega))}, \tag{5}$$

where L is a positive constant independant of p_0.

Remark 1. In this work, the initial condition p_0 is restricted to belong to $L^\infty((0, a_\dagger); L^2(\Omega))$ in order to get a corresponding L^∞ null controllability result - more precisely, in the case when p_0 does not belong to $L^\infty((0, a_\dagger); L^2(\Omega))$, there may not exists a control function in $L^\infty((0, \tau) \times (0, a_\dagger) \times \omega)$ driving p_0 to zero in any arbitrary small time τ.

The null-controllability of the the system modelling age-dependant population dynamics is by now well understood in the case in which diffusion is neglected (see Barbu, Ianelli and Martcheva [12], Hegoburu, Magal, Tucsnak [15], Maity [29], Aniţa and Hegoburu [7]). In the case when spatial diffusion is taken into account, namely for (1), the particular case when the control acts in the whole space (the case corresponding to $\omega = \Omega$) was investigated by S. Aniţa (see [6], p 148). The case when the control acts in a spatial subdomain ω was firstly studied by B. Ainseba [1], where the author proves the null controllability of the above system (1), except for a small interval of ages near zero. The case when the control acts in a spatial subdomain ω and also only for small age classes was investigated by B. Ainseba and S. Aniţa [2], for initial data p_0 in a neighborhood of the target $\widetilde{p}$. As already mentioned, Hegoburu and Tucsnak proved the null controllability of system (1), using an adaptation of the Lebeau Robbiano strategy originally developed for the null-controllability of the heat equation. This result has been recently improved by Maity, Tucsnak and Zuazua [30], assuming that the young individuals are not able to reproduce before some age $a_b > 0$, where the control function u in system (1) has support in some interval of ages $[a_1, a_2]$, where $0 \leqslant a_1 < a_2 \leqslant a_\dagger$. In [30] the authors proved the null controllability result with this additional age restriction, provided that the control time τ is large enough, and the age a_1 is smaller than a_b. Related approximate and exact controllability issues have also been studied in Ainseba and Langlais [4], Ainseba and Iannelli [3], Traore [35], Kavian and Traore [23].

The time optimal control problems for age-structured populations dynamics without diffusion has been extensively studied in the past decades, essentially in the case when the control acts as an harvesting rate (see, for instance, Brokate [13], Barbu and Iannelli [11]). In this case, the bang-bang structure of the optimal harvesting has been obtained in several papers (see, for instance, Medhin [31], Aniţa [5, 6] Aniţa et al [8], Hritonenko and Yatsenko [17], Hritonenko et al [18] and references therein). The literature devoted to the time optimal additive control problems for age-structured populations dynamics with spatial diffusion (namely for system (1)) is less abundant, but several important results and methods are available for the heat equation (see, for instance, Apraiz, Escauriaza, Wang, Zhang [10], Wang [36], Micu, Roventa, Tucsnak [32], Wang and Zhang [37], Apraiz and Escauriaza [9] and references therein). Here, we shall use the strategy developed by Wang [36] (which is, roughly speaking, a generalization of the Lebeau Robianno strategy) with the methodology developed in Hegoburu and Tucsnak [16] (based on the Lebeau Robianno strategy) in order to prove the L^∞ null controllability of system (4) in any time $\tau > 0$ over any subset $E \subset (0,\tau)$ of positive measure. As already claimed, this L^∞ null controllability result implies the bang-bang principle stated in Theorem 1.1.

The remaining part of this work is organized as follows. In Section 2 we introduce some basic results on the Lotka-McKendrick semigroup without spatial diffusion, corresponding to the L^2 and the L^∞ settings, and we state an L^∞ null controllability result associated to system (4) without spatial diffusion. In Section 3 we recall and introduce some results corresponding to the Lotka-McKendrick semigroup with spatial diffusion, associated to the L^2 and the L^∞ settings. Section 4 is devoted to study the L^∞ null controllability of low frequencies for the solution of system (4). We prove the main result in Section 5, by using a strategy developed by Wang [36], originally intented to study the time optimal internal controls for the heat equation.

Notation. In all what follows, C will denote a generic constant, depending only of the coefficients in (1), on Ω and ω, whose value may change from line to line.

2. **Some background on the Lotka-McKendrick semigroup without diffusion.** This section is devoted to recall some existing results on the population semigroup for the linear age-structured model without spatial diffusion relatively to the classical L^2 setting, and to introduce the corresponding results relatively to the L^∞ setting. In particular, we recall the structure of the spectrum of the semigroup generator and we shall state a null controllability result, in the L^∞ setting, concerning the diffusion free case.

2.1. **The free diffusion semigroup in $L^2(0, a_\dagger)$.** In this paragraph we remind some results on the diffusion free case, which is described by the so-called McKendrick-Von Foster model. We do not give proofs and we refer, for instance, to Song et al. [34] or Inaba [21] for a detailed presentation of these issues.

The considered system is:

$$\begin{cases} \partial_t p(t,a) + \partial_a p(t,a) = -\mu(a)p(t,a), & t>0,\ a\in(0,a_\dagger),\\ p(t,0) = \displaystyle\int_0^{a_\dagger} \beta(a)p(t,a)\,\mathrm{d}a, & t>0,\\ p(0,a) = p_0(a), & a\in(0,a_\dagger), \end{cases}$$

where μ and β satisfy the assumptions in Theorem 1.2.

The above system is described by the operator A_0 defined by

$$\mathcal{D}(A_0) = \left\{\varphi \in L^2[0,a_\dagger] \quad | \quad \varphi(0) = \int_0^{a_\dagger} \beta(a)\varphi(a)\,\mathrm{d}a;\ -\frac{\mathrm{d}\varphi}{\mathrm{d}a} - \mu\varphi \in L^2[0,a_\dagger]\right\},$$
$$A_0\varphi = -\frac{\mathrm{d}\varphi}{\mathrm{d}a} - \mu\varphi \qquad (\varphi\in\mathcal{D}(A_0)). \tag{6}$$

Theorem 2.1. *The operator A_0 defined by (6) has compact resolvent and its spectrum is constituted of a countable (infinite) set of isolated eigenvalues with finite algebraic multiplicity. The eigenvalues $(\lambda_n^0)_{n\geqslant 1}$ of A_0 (counted without multiplicity) are the solutions of the characteristic equation*

$$F(\lambda) := \int_0^{a_\dagger} \beta(a) e^{-\lambda a} \pi(a)\, \mathrm{d}a = 1. \tag{7}$$

The eigenvalues $(\lambda_n^0)_{n\geqslant 1}$ are of geometric multiplicity one, the eigenspace associated to λ_n^0 being the one-dimensional subspace of $L^2(0, a_\dagger)$ generated by the function

$$\varphi_n^0(a) = e^{-\lambda_n^0} \pi(a) = e^{-\lambda_n^0 a - \int_0^a \mu(s)\, \mathrm{d}s}.$$

Finally, every vertical strip of the complex plane $\alpha_1 \leqslant \mathrm{Re}(z) \leqslant \alpha_2$, $\alpha_1, \alpha_2 \in \mathbb{R}$, contains a finite number of eigenvalues of A_0.

Theorem 2.2. *The operator A_0 defined by (6) has a unique real eigenvalue λ_1^0. Moreover, we have the following properties :*

1. *λ_1^0 is of algebraic multiplicity one;*
2. *$\lambda_1^0 > 0$ (resp. $\lambda_1^0 < 0$) if and only if $F(0) > 1$ (resp. $F(0) < 1$);*
3. *λ_1^0 is a real dominant eigenvalue:*

$$\lambda_1^0 > \mathrm{Re}(\lambda_n^0), \qquad \forall n \geqslant 2.$$

It is well known (see, for instance, Song et al. [34] or Kappel and Zhang [22]) that A generates a C^0 semigroup of linear operators in $L^2[0, a_\dagger]$ which we denote by $\mathbb{T}^{A_0} = (\mathbb{T}_t^{A_0})_{t\geqslant 0}$. We also have the following useful result (see, for instance, [21, p 23]):

Proposition 1. *The semigroup $\mathbb{T}^{A_0}$ generated on $L^2([0, a_\dagger])$ by A_0 is compact for $t \geqslant a_\dagger$.*

According to Zabczyk [39, Section 2]), this implies in particular that

$$\omega_a(A_0) = \omega_0(A_0),$$

where $\omega_a(A_0) := \lim\limits_{t\to+\infty} t^{-1} \ln \|\mathbb{T}_t^{A_0}\|_{L^2(0,a_\dagger)}$ denotes the growth bound of the semigroup $\mathbb{T}_t^{A_0}$ and $\omega_0(A_0) := \sup\{\mathrm{Re}\lambda \mid \lambda \in \sigma(A_0)\}$ the spectral bound of A_0. It is worth noticing that the above condition ensures that the exponential stability of $\mathbb{T}^{A_0}$ is equivalent to the condition $\omega_0(A_0) < 0$. According to Theorem 2.1 and 2.2, it follows that the exponential stability of $\mathbb{T}^{A_0}$ is equivalent to the condition $\lambda_1^0 < 0$, where $\lambda_1^0 < 0$ is the unique real solution to the characteristic equation defined by (7).

The following subsection is intended to show that the condition $\lambda_1^0 < 0$ is also sufficient to get a stability type result in the L^∞ setting.

2.2. **About the diffusion free semigroup in $L^\infty(0, a_\dagger)$.** This subsection is devoted to discuss some properties of the Lotka-McKendrick semigroup $\mathbb{T}^{A_0}$ without diffusion, in the case when the initial inputs are restricted to belong to the state space $L^\infty(0, a_\dagger)$. More precisely, we shall introduce an L^∞ exponential stability type result, and state an L^∞ null-controllability result associated to system (4) in the diffusion free case.

The following Lemma 2.3 states that the condition $\lambda_1^0 < 0$ (where $\lambda_1^0 < 0$ is the unique real solution to the characteristic equation defined by (7)) is sufficient to get an L^∞ exponential stability type result:

Lemma 2.3. *Let λ_1^0 be the unique real solution to the characteristic equation defined by (7). There exists a constant $C \geqslant 0$ such that for every $p_0 \in L^\infty(0, a_\dagger)$, we have*

$$|\mathbb{T}_t^{A_0} p_0(a)| \leqslant \begin{cases} |p_0(a-t)| & \text{if } a \geqslant t, \\ Ce^{\lambda_1^0(t-a)}\|p_0\|_{L^1(0,a_\dagger)} & \text{if } t > a. \end{cases} \tag{8}$$

Proof. Let $p_0 \in L^\infty(0, a_\dagger)$. It is well known (see, for instance, Iannelli [20] or Webb [38]) that the semigroup $\mathbb{T}^{A_0}$ satisfies

$$(\mathbb{T}_t^{A_0} p_0)(a) = \begin{cases} \dfrac{\pi(a)}{\pi(a-t)} p_0(a-t) & \text{if } \ t \leqslant a, \\ \pi(a) b(t-a) & \text{if } \ t > a, \end{cases} \tag{9}$$

where $\pi(a) = e^{-\int_0^a \mu(\sigma)\,\mathrm{d}\sigma}$ is the probability of survival of an individual from age 0 to a and $b(t) = (\mathbb{T}_t^{A_0} p_0)(0) = \displaystyle\int_0^{a_\dagger} \beta(a)(\mathbb{T}_t^{A_0} p_0)(a)\,\mathrm{d}a$ is the total birth rate function.

If $a \geqslant t$, it is clear that we have $|\mathbb{T}_t^{A_0} p_0(a)| \leqslant |p_0(a-t)|$ due to (9), so that the first estimate of (8) holds.

Let $s > 0$. It is shown in Iannelli [20, p. 21 and 22] (see also Aniţa [6, p. 54]) that for every $s > 0$ we may write the total birth rate $b(s)$ in the following form:

$$b(s) = e^{\lambda_1^0 s}(b_0 + e^{-\lambda_1^0 s} F(s) + \Omega_0(s)) \qquad (s > 0), \tag{10}$$

where b_0 is a nonnegative constant satisfying

$$0 \leqslant b_0 \leqslant M_0 \|p_0\|_{L^1(0,a_\dagger)}, \tag{11}$$

the function $\Omega_0 \in L^\infty(0, +\infty)$ satisfies, for every $s > 0$,

$$|\Omega_0(s)| \leqslant M_0 \|p_0\|_{L^1(0,a_\dagger)}, \tag{12}$$

and the map $F \in L^\infty(0, +\infty)$ is defined by

$$F(s) := \begin{cases} \displaystyle\int_s^{a_\dagger} \beta(x) \frac{\pi(x)}{\pi(x-s)} p_0(x-s)\,\mathrm{d}x & \text{if } \ s \in [0, a_\dagger], \\ 0 & \text{if } \ s > a_\dagger, \end{cases} \tag{13}$$

where the constant M_0 given in the inequalities (11) and (12) do not depend on p_0. It follows from (10), (11), (12) and (13) that for every $s > 0$ we have

$$|b(s)| \leqslant e^{\lambda_1^0 s}(M_0\|p_0\|_{L^1(0,a_\dagger)} + (\sup_{\sigma\in[0,a_\dagger]} e^{-\lambda_1^0 \sigma}) \cdot \|\beta\|_{L^\infty(0,a_\dagger)} \|p_0\|_{L^1(0,a_\dagger)} + M_0\|p_0\|_{L^1(0,a_\dagger)}),$$

so that the second estimate of (8) follows from the above inequality, setting

$$C := 3 \max\left(M_0 \ ; \ (\sup_{\sigma\in[0,a_\dagger]} e^{-\lambda_1^0 \sigma}) \cdot \|\beta\|_{L^\infty(0,a_\dagger)}\right).$$

□

Recall that the main motivation of this paper is to show that the system (4) is null controllable in any time $\tau > 0$ with controls $u \in L^\infty((0,\tau) \times (0, a_\dagger) \times \omega)$ which are additionnaly supported in time in an arbitrary subset $E \subset (0, \tau)$ of positive measure. To this aim, it will be needed - in the following Section 4 - to get a corresponding null controllability result in the diffusion free case. More precisely, the diffusion free control problem associated to system (4) writes as

$$\begin{cases} \partial_t p(t,a) + \partial_a p(t,a) + \mu(a) p(t,a) = v(t,a)\chi_E(t) & t \in (0,\tau),\ a \in (0,a_\dagger), \\ p(t,0) = \displaystyle\int_0^{a_\dagger} \beta(a) p(t,a)\,\mathrm{d}a, & t \in (0,\tau), \\ p(0,a) = p_0(a), & a \in (0,a_\dagger), \end{cases} \tag{14}$$

where v denotes the control function and E is a subset of $[0,\tau]$ with positive measure. Let us state an L^∞ null-controllability result for the above system (14):

Proposition 2. *Under the above assumptions, let $\tau > 0$ and E be a subset of $[0,\tau]$ with positive measure, i.e. $m(E) > 0$. Then for every $p_0 \in L^\infty(0,a_\dagger)$, there exists $v \in L^\infty((0,\tau)\times(0,a_\dagger))$ such that the solution p of (14) satisfies*

$$p(\tau,a) = 0 \qquad (a \in (0,a_\dagger) \text{ a.e.}).$$

Moreover, for almost every $(t,a) \in (0,\tau)\times(0,a_\dagger)$, we have

$$|v(t,a)| \leqslant \frac{1}{m(E)} |\mathbb{T}_t^{A_0} p_0(a)| \qquad ((t,a) \in (0,\tau)\times(0,a_\dagger) \ \text{ a.e.}). \tag{15}$$

Proof. Let $\tau > 0$. For almost every $(t,a) \in (0,\tau)\times(0,a_\dagger)$, we set

$$v(t,a) = -\frac{1}{m(E)} (\mathbb{T}_t^{A_0} p_0)(a) \quad (t \in (0,\tau),\ a \in (0,a_\dagger)). \tag{16}$$

With v defined as above, the (mild) solution p of (14) satisfies

$$\begin{aligned} p(t,a) &= \mathbb{T}_t^{A_0} p_0(a) + \int_0^t (\mathbb{T}_{t-\sigma}^{A_0} \chi_E(\sigma) v(\sigma))(a)\,\mathrm{d}\sigma \\ &= \mathbb{T}_t^{A_0} p_0(a) - \frac{1}{m(E)} \int_0^t \chi_E(\sigma) \mathbb{T}_{t-\sigma}^{A_0} \left(\mathbb{T}_\sigma^{A_0} p_0\right)(a)\,\mathrm{d}\sigma \\ &= \mathbb{T}_t^{A_0} p_0(a) - \frac{1}{m(E)} \int_0^t \chi_E(\sigma) \mathbb{T}_t^{A_0} p_0(a)\,\mathrm{d}\sigma \\ &= \mathbb{T}_t^{A_0} p_0(a) \left(1 - \frac{m(E\cap(0,t))}{m(E)}\right), \end{aligned}$$

so that we have $p(\tau,\cdot) = 0$ since E is a subset of $(0,\tau)$. The estimate (15) is then a direct consequence of (16). □

3. **The population dynamics with diffusion.** This section is devoted to recall and introduce some results concerning the Lotka-McKendrick semigroup with spatial diffusion, in both L^2 and L^∞ settings. More precisely, we recall the structure of the spectrum of the semigroup generator and we give a stability result in both L^2 and L^∞ settings, conditionally to the sign of λ_1^0 (where λ_1^0 denotes the unique real solution to (7)).

3.1. **The Lotka-McKendrick semigroup with diffusion in $L^2((0,a_\dagger)\times\Omega)$.** The existence of a semigroup on $L^2((0,a_\dagger)\times\Omega)$ describing the linear age-structured population model with diffusion coefficient and age dependent birth and death rates, with homogeneous Neumann boundary conditions has been proved in Huyer [19, Theorem 2.8] (see also Guo and Chan [14] for the case of homogeneous Dirichlet boundary conditions).

More precisely, let $H := L^2((0,a_\dagger)\times\Omega)$ and let us consider the diffusion free population operator $A_1 : \mathcal{D}(A_1)\to H$ defined by

$$\mathcal{D}(A_1) = \Big\{\varphi\in H \mid \varphi(\cdot,x) \text{ is locally absolutely continuous on } [0,a_\dagger), \\ \varphi(0,x) = \int_0^{a_\dagger}\beta(a)\varphi(a,x)\,\mathrm{d}a \text{ for a.e. } x\in\Omega,\ \frac{\partial\varphi}{\partial a}+\mu\varphi\in H\Big\},$$

$$A_1\varphi = -\frac{\partial\varphi}{\partial a}-\mu\varphi,$$

and the diffusion operator $A_2 : \mathcal{D}(A_2)\to H$ defined by

$$\mathcal{D}(A_2) = \Big\{\varphi\in H \mid \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \partial\Omega\Big\},\ A_2\varphi = \Delta\varphi.$$

The population operator with diffusion $A : \mathcal{D}(A)\to H$ is defined by

$$\mathcal{D}(A) = \mathcal{D}(A_1)\cap\mathcal{D}(A_2),\quad A\varphi = A_1\varphi + A_2\varphi.$$

The generator A of the population semigroup can be seen as the sum of a population operator without diffusion $-\mathrm{d}/\mathrm{d}a-\mu I$ and a spatial diffusion term Δ. It turns out that spectral properties of A can be easily obtained from those of these two operators.

Theorem 3.1. *Let $0=\lambda_0<\lambda_1\leqslant\lambda_2\leqslant\dots$ be the increasing sequence of eigenvalues of $-\Delta$ with Neumann boundary conditions and let $(\varphi_n)_{n\geqslant 0}$ be a corresponding orthonormal basis of $L^2(\Omega)$. Let $(\lambda_n^0)_{n\geqslant 1}$ and $(\varphi_n^0)_{n\geqslant 1}$ be respectively the sequence of eigenvalues and eigenfunctions of the free diffusion operator A_0 defined by (6) (see Theorem 2.1). Then the following assertions hold:*

1. *The eigenvalues of A are given by*
$$\sigma(A) = \{\lambda_i^0-\lambda_j \mid i\in\mathbb{N}^*,\ j\in\mathbb{N}\}.$$
2. *A has a dominant eigenvalue:*
$$\lambda_1 = \lambda_1^0 > \mathrm{Re}(\lambda),\qquad \forall\lambda\in\sigma(A),\ \lambda\neq\lambda_1.$$
3. *The eigenspace associated to an eigenvalue λ of A is given by*
$$\mathrm{Span}\{\varphi_i^0(a)\varphi_j(x) = e^{-\lambda_i^0 a}\pi(a)\varphi_j(x) \mid \lambda_i^0-\lambda_j=\lambda\}.$$

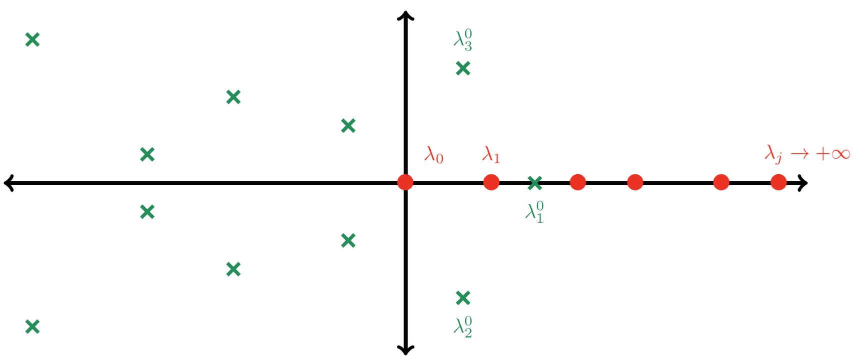

FIGURE 1. The spectrum of the free diffusion operator A_0 (green crosses) and of $-\Delta$ (red circles)

Since the operator A generates a $\mathcal{C}^0$ semigroup of linear operators in H which we denote by $\mathbb{T}^A = (\mathbb{T}^A_t)_{t\geqslant 0}$, this allows to define the concept of (mild) solution of (4) in the following standard way: we say that p is a mild solution of (4) if

$$p(t,\cdot) = \mathbb{T}^A_t p_0 + \Phi_{t,E} u \qquad (t \geqslant 0, \quad u \in L^2([0,\infty); H)), \tag{17}$$

where the control operator $B \in \mathcal{L}(H)$ is defined by

$$Bu = \chi_\omega u \qquad (u \in H),$$

and where

$$\Phi_{t,E} u = \int_0^t \mathbb{T}^A_{t-\sigma} B \chi_E(\sigma) u(\sigma)\,\mathrm{d}\sigma \qquad (t \geqslant 0, \quad u \in L^2([0,\infty); H)). \tag{18}$$

It is worth noticing, for instance by using a spectral decomposition, that the semigroup $\mathbb{T}^A$ is exponentially stable if $\lambda_1^0 < 0$, where we recall that λ_1^0 denotes the unique real solution to the characteristic equation defined by (7).

Remark 2. In order to prove the null controllability of system (4), we may assume, without loss of generality, that the so called reproductive number satisfies

$$\int_0^{a_\dagger} \beta(a)\pi(a)\,\mathrm{d}a < 1,$$

which implies that the unique real solution λ_1^0 to the characteristic equation defined by (7) satisfies $\lambda_1^0 < 0$. Indeed, in the case when $\int_0^{a_\dagger} \beta(a)\pi(a)\,\mathrm{d}a \geqslant 1$, we may consider the auxiliary system

$$\begin{cases} \partial_t z(t,a,x) + \partial_a z(t,a,x) + \widetilde{\mu}(a) z(t,a,x) - \Delta z(t,a,x) \\ \qquad = \chi_\omega(x)\chi_E(t) v(t,a,x), & t>0,\ a\in(0,a_\dagger),\ x\in\Omega, \\ \dfrac{\partial z}{\partial \nu}(t,a,x) = 0, & t>0,\ a\in(0,a_\dagger),\ x\in\partial\Omega, \\ z(t,0,x) = \displaystyle\int_0^{a_\dagger} \beta(a) z(t,a,x)\,\mathrm{d}a, & t>0,\ x\in\Omega, \\ z(0,a,x) = p_0(a,x), & a\in(0,a_\dagger),\ x\in\Omega, \end{cases} \tag{19}$$

with $\widetilde{\mu}(a) := \mu(a) + \lambda$, where $\lambda \geqslant 0$ is large enough to have

$$\int_0^{a_\dagger} \beta(a) e^{-\int_0^a \widetilde{\mu}(s)\,\mathrm{d}s}\,\mathrm{d}a < 1.$$

Suppose that the above system (19) is null controllable with control function v. Then, system (4) is null controllable with control function $u = e^{\lambda t} v$, which has the same regularity as v.

From now on, without loss of generality (see the above Remark 2), we assume that the unique real solution λ_1^0 to the characteristic equation defined by (7) satisfies the following assumption:

(Stable) : the unique real solution λ_1^0 to the characteristic equation (7) satisfies $\lambda_1^0 < 0$.

Recall that the above assumption gives the stability of the semigroup $\mathbb{T}^A$ in the space $H = L^\infty((0,a_\dagger); L^2(\Omega))$. The aim of the following subsection is to prove that the above assumption **(Stable)** also induces some stability type results in the subspace $L^\infty((0,a_\dagger); L^2(\Omega))$ of H.

3.2. **Stability results in $L^\infty((0,a_\dagger);L^2(\Omega))$.** In order to derive from the above subsection some stability results in the L^∞ setting, let us recall from Theorem 3.1 that $\{\varphi_j\}_{j\geqslant 0}$ denotes an orthonormal basis in $L^2(\Omega)$ formed of eigenvectors of the Neumann Laplacian and $(\lambda_j)_{j\geqslant 0}$ is the corresponding non decreasing sequence of eigenvalues. In other words $(\varphi_j)_{j\geqslant 0}$ is an orthonormal basis in $L^2(\Omega)$ such that for every $j\geqslant 0$ we have

$$\begin{cases} -\Delta\varphi_j = \lambda_j\varphi_j & \text{in } \Omega,\\ \dfrac{\partial\varphi_j}{\partial\nu} = 0 & \text{on } \partial\Omega.\end{cases}$$

The following Lemma 3.2 states that the condition $\lambda_1^0 < 0$ (where λ_1^0 is the unique real solution the characteristic equation defined by (7)) ensures some stability results in the space $L^\infty((0,a_\dagger);L^2(\Omega))$. More precisely, we have

Lemma 3.2. *Under the assumption* **(Stable)**, *there exists a constant $C\geqslant 0$ such that for every $p_0\in L^\infty((0,a_\dagger);L^2(\Omega))$, for every $(t,a)\in(0,+\infty)\times(0,a_\dagger)$ we have*

$$\sum_{j=0}^{+\infty}(\mathbb{T}_t^{A_0}p_0^j(a))^2 \leqslant C\|p_0\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))} \qquad (t\geqslant 0,\ a\in(0,a_\dagger)\ a.e.), \tag{20}$$

and for every $t\geqslant 0$ we have

$$\|\mathbb{T}_t^{A}p_0\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))} \leqslant C\|p_0\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))} \qquad (t\geqslant 0), \tag{21}$$

where $p_0^j(a) := (p_0(a,\cdot),\varphi_j)_{L^2(\Omega)}$ for almost every $a\in(0,a_\dagger)$ and every integer $j\geqslant 0$.

Proof. Let $p_0\in L^\infty((0,a_\dagger)\times\Omega))$. For every $j\geqslant 0$ and for almost every $a\in(0,a_\dagger)$, denote by $p_0^j(a) = (p_0(a,\cdot),\varphi_j)_{L^2(\Omega)}$. Note that, for almost every $a\in(0,a_\dagger)$ we have

$$\sum_{j=0}^{+\infty}(p_0^j(a))^2 = \int_\Omega p_0(a,x)^2\,\mathrm{d}x, \tag{22}$$

as a consequence of Parseval's formula.

Let $a\geqslant t$. Using (8) together with (22), we have

$$\sum_{j=0}^{+\infty}(\mathbb{T}_t^{A_0}p_0^j(a))^2 \leqslant \sum_{j=0}^{+\infty}(p_0^j(a-t))^2 = \int_\Omega p_0(a-t,x)^2\,\mathrm{d}x \leqslant \|p_0\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))}. \tag{23}$$

If $t\geqslant a$, since $\lambda_1^0\leqslant 0$, from (8) we have

$$\begin{aligned}\sum_{j=0}^{+\infty}(\mathbb{T}_t^{A_0}p_0^j(a))^2 &\leqslant C\sum_{j=0}^{+\infty}\|p_0^j\|^2_{L^1(0,a_\dagger)}\\ &\leqslant C\sum_{j=0}^{+\infty}\|p_0^j\|^2_{L^2(0,a_\dagger)}\\ &= C\|p_0\|^2_{L^2((0,a_\dagger);L^2(\Omega))}\\ &\leqslant C\|p_0\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))},\end{aligned} \tag{24}$$

so that the estimation (20) follows from (23) and (24).

In order to get the estimation (21), notice that for almost every $(t,a) \in (0,+\infty)\times(0,a_\dagger)$ we have

$$\mathbb{T}_t^A p_0(a,\cdot) = \sum_{j=0}^{+\infty} p^j(t,a)\varphi_j \ \text{ in } L^2(\Omega), \ \text{a.e. in } (0,+\infty)\times(0,a_\dagger),$$

where

$$\begin{cases} \partial_t p^j(t,a) + \partial_a p^j(t,a) + (\mu(a)+\lambda_j)p^j(t,a) = 0 & t>0,\ a\in(0,a_\dagger), \\ p^j(t,0) = \displaystyle\int_0^{a_\dagger} \beta(a)p^j(t,a)\,\mathrm{d}a, & t>0, \\ p^j(0,a) = p_0^j(a), & a\in(0,a_\dagger). \end{cases} \tag{25}$$

It is easy to check that the solution p^j of (25) satisfies $p^j(t,a) = e^{-t\lambda_j}\mathbb{T}_t^{A_0}(p_0^j)(a)$, so that for almost every $(t,a)\in(0,+\infty)\times(0,a_\dagger)$ we have

$$\|\mathbb{T}_t^A p_0(a,\cdot)\|^2_{L^2(\Omega)} = \sum_{j=0}^{+\infty}|p^j(t,a)|^2 = \sum_{j=0}^{+\infty} e^{-2t\lambda_j}(\mathbb{T}_t^{A_0}p_0^j(a))^2 \leqslant \sum_{j=0}^{+\infty}(\mathbb{T}_t^{A_0}p_0^j(a))^2,$$

and the estimation (21) follows from (20) and the above inequality. □

4. Low frequency control in $L^\infty((0,a_\dagger);L^2(\Omega))$.

In this section, we prove that the projection of the state trajectory of (4) on an infinite subspace of $H = L^2((0,a_\dagger);L^2(\Omega))$ (defined using the eigenfunctions of the Neumann Laplacian) can be steered to zero in any time and we estimate the norm of the associated control.

In the sequel, for any $\mu>0$, we denote by

$$\mathcal{N}(\mu) := \mathrm{Card}\{k : \lambda_k \leqslant \mu\},$$

$$E_\mu := \mathrm{Span}\{\varphi_k \ : \lambda_k \leqslant \mu\},$$

and $\Pi_{E_\mu} : L^2(\Omega)\to L^2(\Omega)$ the orthogonal projection from $L^2(\Omega)$ onto E_μ. The main result of this section is:

Proposition 3. *Under the assumption* **(Stable)**, *let $\mu>0$ and let $T>0$. Let E be a subset of positive measure in the interval $[0,T]$. There exists $u_\mu \in L^\infty((0,T)\times(0,a_\dagger)\times\omega)$ such that the solution p of (4) satisfies*

$$\Pi_{E_\mu} p(T,a,\cdot) = 0 \qquad (a\in(0,a_\dagger) \text{ a.e.}).$$

Moreover, we have the following estimate:

$$\|u_\mu\|^2_{L^\infty((0,T)\times(0,a_\dagger)\times\omega)} \leqslant \frac{C_1 e^{C_2\sqrt{\mu}}}{(m(E))^2}\|p_0\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))},$$

where C_1 and C_2 are two positive constants.

The main ingredient of the proof is an inequality involving the eigenfunctions of the Neumann Laplacian:

Theorem 4.1. *For any non-empty open subset ω of Ω, there exists two positive constants $C_1, C_2 \geqslant 0$ such that for any $\mu>0$, for any sequence $(a_j)_{j\geqslant 0}\subset\mathbb{R}$, we have*

$$\left(\sum_{j:\lambda_j\leqslant\mu}|a_j|^2\right)^{1/2} \leqslant C_1 e^{C_2\sqrt{\mu}} \int_\omega \left|\sum_{j:\lambda_j\leqslant\mu} a_j\varphi_j(x)\right| \mathrm{d}x. \tag{26}$$

The above inequality may be obtained by combining results and methods from Lebeau and Robbiano [24], [25] (see also Lü [27]), and analyticity arguments due to Apraiz et al. [10, Proof of Theorem 5]. Besides the above inequality, we will use a classical duality argument, following the methodology in Micu, Roventa and Tucsnak [32] - roughly speaking, the L^1 observability induces the L^∞ controllability.

Proof of Proposition 3. Note that the solution p of (4) writes

$$p(t,a,\cdot) = \sum_{j=0}^{+\infty} p^j(t,a)\varphi_j \ \text{ in } L^2(\Omega), \ \text{a.e. in } (0,T)\times(0,a_\dagger),$$

where

$$\begin{cases} \partial_t p^j(t,a) + \partial_a p^j(t,a) + (\mu(a)+\lambda_j)p^j(t,a) \\ \qquad = \chi_E(t)\displaystyle\int_\omega u(t,a,x)\varphi_j(x)\,\mathrm{d}x, & t\in(0,T),\ a\in(0,a_\dagger), \\ p^j(t,0) = \displaystyle\int_0^{a_\dagger}\beta(a)p^j(t,a)\,\mathrm{d}a, & t\in(0,T), \\ p^j(0,a) = p_0^j(a), & a\in(0,a_\dagger), \end{cases}$$

and where

$$p_0(a,\cdot) = \sum_{j=0}^{+\infty} p_0^j(a)\varphi_j \ \text{ in } L^2(\Omega), \ \text{a.e. } a\in(0,a_\dagger).$$

The aim is to solve the following moment problem: find $u \in L^\infty((0,T)\times(0,a_\dagger)\times\omega)$ such that for every $j\in[0,\mathcal{N}(\mu)]$, we have

$$\int_\omega u_\mu(t,a,x)\varphi_j(x)\,\mathrm{d}x = v_j(t,a) \qquad ((t,a)\in(0,T)\times(0,a_\dagger) \ \text{ a.e.}),$$

where v_j denotes a null control associated to the system

$$\begin{cases} \partial_t p^j(t,a) + \partial_a p^j(t,a) + (\mu(a)+\lambda_j)p^j(t,a) \\ \qquad = \chi_E(t)v_j(t,a), & t\in(0,T),\ a\in(0,a_\dagger), \\ p^j(t,0) = \displaystyle\int_0^{a_\dagger}\beta(a)p^j(t,a)\,\mathrm{d}a, & t\in(0,T), \\ p^j(0,a) = p_0^j(a), & a\in(0,a_\dagger). \end{cases} \tag{27}$$

Recall from Proposition 2 that for every $j \geqslant 0$, there exists $v_j \in L^\infty((0,T)\times(0,a_\dagger))$ such that the corresponding solution p^j of the above system (27) satisfies $p^j(T,a)=0$ for almost every $a\in(0,a_\dagger)$, with $|v_j(t,a)| \leqslant \frac{1}{m(E)}|(\mathbb{T}_t^{A_0}p_0^j(a))|$ (since we can choose $v_j(t,a) = -\frac{e^{-t\lambda_j}}{m(E)}(\mathbb{T}_t^{A_0}p_0^j)(a))$.

Let $\mu>0$. Define the map $\mathcal{G} : L^2(\omega)\to\mathbb{R}^{\mathcal{N}(\mu)+1}$ by

$$\mathcal{G}u := \left(\int_\omega u(x)\varphi_j(x)\,\mathrm{d}x\right)_{0\leqslant j\leqslant\mathcal{N}(\mu)}.$$

It is easy to check that for every $w=(w_j)_{0\leqslant j\leqslant\mathcal{N}(\mu)}\in\mathbb{R}^{\mathcal{N}(\mu)+1}$, we have

$$\mathcal{G}^\star w = \sum_{j=0}^{\mathcal{N}(\mu)} w_j\varphi_j.$$

Inequality (26) ensures that for every $w \in \mathbb{R}^{\mathcal{N}(\mu)+1}$, we have

$$\|w\|_{\mathbb{R}^{\mathcal{N}(\mu)+1}} \leqslant C_1 e^{C_2\sqrt{\mu}} \|\mathcal{G}^\star w\|_{L^1(\omega)}. \tag{28}$$

Define the mapping $\mathcal{K} : E_\mu \to \mathbb{R}^{\mathcal{N}(\mu)+1}$ by the formula $\mathcal{K}(\mathcal{G}^\star w) = w$ for all $w \in \mathbb{R}^{\mathcal{N}(\mu)+1}$. Using (28), it follows that the mapping $\mathcal{K}$ is well defined and that

$$\|\mathcal{K}z\|_{\mathbb{R}^{\mathcal{N}(\mu)+1}} \leqslant C_1 e^{C_2\sqrt{\mu}} \|z\|_{L^1(\omega)} \qquad (z \in E_\mu). \tag{29}$$

For every $i \in \{0, \dots, \mathcal{N}(\mu)\}$, define the linear mapping $\mathcal{K}_i : E_\mu \to \mathbb{R}$ by the formula $\mathcal{K}_i(z) = (\mathcal{K}(z))_i$ for all $z \in E_\mu$, where $(\mathcal{K}(z))_i$ denotes the i-th component of the vector $\mathcal{K}(z)$. It follows from (29) that, for every $i \in \{0, \dots, \mathcal{N}(\mu)\}$ and for every $z \in E_\mu$ we have

$$|\mathcal{K}_i(z)| \leqslant C_1 e^{C_2\sqrt{\mu}} \|z\|_{L^1(\omega)} \qquad (i \in \{0, \dots, \mathcal{N}(\mu)\},\ z \in E_\mu),$$

so that from the Hahn-Banach theorem we can extend each linear functional $\mathcal{K}_i$ to a bounded linear functional $\widetilde{\mathcal{K}}_i$ on $L^1(\omega)$ such that

$$|\widetilde{\mathcal{K}}_i(z)| \leqslant C_1 e^{C_2\sqrt{\mu}} \|z\|_{L^1(\omega)} \qquad (i \in \{0, \dots, \mathcal{N}(\mu)\},\ z \in L^1(\omega)). \tag{30}$$

Now, let us define the mapping $\widetilde{\mathcal{K}} : L^1(\omega) \to \mathbb{R}^{\mathcal{N}(\mu)+1}$ by the formula $\widetilde{\mathcal{K}}(z) := (\widetilde{\mathcal{K}}_i(z))_{0\leqslant i\leqslant \mathcal{N}(\mu)}$ for all $z \in L^1(\omega)$, so that the mapping $\widetilde{\mathcal{K}}$ is a bounded linear extension of the mapping $\mathcal{K}$ on $L^1(\omega)$. Notice that, using (30), for every $z \in L^1(\omega)$ we have

$$\|\widetilde{\mathcal{K}}z\|_{\mathbb{R}^{\mathcal{N}(\mu)+1}} \leqslant C_1 e^{C_2\sqrt{\mu}} (\mathcal{N}(\mu)+1)^{\frac{1}{2}} \|z\|_{L^1(\omega)} \qquad (z \in L^1(\omega)).$$

By Weyl's formula (see, for instance, Netrusov and Safarov [33] for a reminder), there exists a constant $K > 0$ such that $\mathcal{N}(\mu) \leqslant K\mu^{\frac{N}{2}}$, so that we may infer from the above estimation that

$$\|\widetilde{\mathcal{K}}z\|_{\mathbb{R}^{\mathcal{N}(\mu)+1}} \leqslant C_1 e^{C_2\sqrt{\mu}} \|z\|_{L^1(\omega)} \qquad (z \in L^1(\omega)), \tag{31}$$

for some other constants $C_1, C_2 \geqslant 0$.

Note that for every $w \in \mathbb{R}^{\mathcal{N}(\mu)+1}$, since $\mathcal{G}^\star w \in E_\mu$ we have

$$\widetilde{\mathcal{K}}(\mathcal{G}^\star w) = \widetilde{\mathcal{K}}|_{E_\mu}(\mathcal{G}^\star w) = \mathcal{K}(\mathcal{G}^\star w) = w,$$

so that $\widetilde{\mathcal{K}}\mathcal{G}^\star = \mathrm{Id}_{\mathcal{L}(\mathbb{R}^{\mathcal{N}(\mu)+1})}$ and since $\widetilde{\mathcal{K}}$ and $\mathcal{G}$ are bounded opeartors, it follows that we have

$$\mathcal{G}\left(\widetilde{\mathcal{K}}\right)^\star = \mathrm{Id}_{\mathcal{L}(\mathbb{R}^{\mathcal{N}(\mu)+1})}. \tag{32}$$

Noting that the range of $\left(\widetilde{\mathcal{K}}\right)^\star$ is included in $(L^1(\omega))' = L^\infty(\omega)$ (from the Riesz representation theorem), it follows from the above equality (32) that for every $w \in \mathbb{R}^{\mathcal{N}(\mu)+1}$, there exists $u := \left(\widetilde{\mathcal{K}}\right)^\star(w) \in L^\infty(\omega)$ such that $\mathcal{G}u = w$, with

$$\begin{aligned}
\|u\|_{L^\infty(\omega)} &= \left\|\left(\widetilde{\mathcal{K}}\right)^\star(w)\right\|_{L^\infty(\omega)} \\
&\leqslant \left\|\left(\widetilde{\mathcal{K}}\right)^\star\right\|_{\mathcal{L}(\mathbb{R}^{\mathcal{N}(\mu)+1}, L^\infty(\omega))} \|w\|_{\mathbb{R}^{\mathcal{N}(\mu)+1}} \\
&= \left\|\widetilde{\mathcal{K}}\right\|_{\mathcal{L}(L^1(\omega), \mathbb{R}^{\mathcal{N}(\mu)+1})} \|w\|_{\mathbb{R}^{\mathcal{N}(\mu)+1}} \\
&\leqslant C_1 e^{C_2\sqrt{\mu}} \|w\|_{\mathbb{R}^{\mathcal{N}(\mu)+1}}.
\end{aligned}$$

in the above inequalities, we have used that $\left\| \left(\widetilde{\mathcal{K}}\right)^\star \right\|_{\mathcal{L}(\mathbb{R}^{\mathcal{N}(\mu)+1},L^\infty(\omega))} = \left\|\widetilde{\mathcal{K}}\right\|_{\mathcal{L}(L^1(\omega),\mathbb{R}^{\mathcal{N}(\mu)+1})}$ together with the estimation $\left\|\widetilde{\mathcal{K}}\right\|_{\mathcal{L}(L^1(\omega),\mathbb{R}^{\mathcal{N}(\mu)+1})} \leqslant C_1 e^{C_2\sqrt{\mu}}$ which follows from (31).

Let $(t,a) \in (0,T)\times(0,a_\dagger)$. Setting $w(t,a) := (v_j(t,a))_{0\leqslant j\leqslant \mathcal{N}(\mu)}$ where v_j is the null control defined by Proposition 2 (with $\mu(a)$ replaced by $(\mu(a)+\lambda_j)$), it follows that there exists $u_\mu(t,a,\cdot)\in L^\infty(\omega)$ such that $Gu_\mu(t,a) = w(t,a)$, *i.e.*

$$\int_\omega u_\mu(t,a,x)\varphi_j(x)\,\mathrm{d}x = v_j(t,a) \qquad (j\in[|0,\mathcal{N}(\mu)|],\ (t,a)\in(0,T)\times(0,a_\dagger)\ \text{ a.e.}),$$

with

$$\|u_\mu(t,a,\cdot)\|_{L^\infty(\omega)} \leqslant C_1 e^{C_2\sqrt{\mu}} \|w(t,a)\|_{\mathbb{R}^{\mathcal{N}(\mu)+1}}. \tag{33}$$

From the above inequality (33), it follows that for almost every $(t,a)\in(0,T)\times(0,a_\dagger)$, we have

$$\|u_\mu(t,a,\cdot)\|_{L^\infty(\omega)} \leqslant C_1 e^{C_2\sqrt{\mu}} \left(\sum_{j=0}^{\mathcal{N}(\mu)} |v_j(t,a)|^2\right)^{1/2},$$

where $|v_j(t,a)|^2 \leqslant \frac{(\mathbb{T}_t^{A_0} p_0^j(a))^2}{(m(E))^2}$ by Proposition 2, so that using (20) together with the above inequality we get that

$$\|u_\mu(t,a,\cdot)\|^2_{L^\infty(\omega)} \leqslant C\times \frac{C_1^2 e^{2C_2\sqrt{\mu}}}{(m(E))^2}\|p_0\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))} \qquad ((t,a)\in(0,T)\times(0,a_\dagger)\ \text{ a.e.}),$$

and we deduce from the above inequality (recalling the constant $C\times C_1^2$ by C_1 and recalling $2C_2$ by C_2) that we have

$$\|u_\mu\|^2_{L^\infty((0,T)\times(0,a_\dagger)\times\omega)} \leqslant \frac{C_1 e^{C_2\sqrt{\mu}}}{(m(E))^2}\|p_0\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))}.$$

□

5. **Proof of the main result.** In this section we prove Theorem 1.2 by using a slight adaptation of the strategy developed by Wang [36], initially proposed to study the time optimal internal control problem for the heat equation.

First, recall from Proposition 3 that, given a time $T>0$, a subset $E\subset(0,T)$ of positive measure and a cutting frequency $\mu>0$, there exists $u_\mu\in L^\infty((0,T)\times(0,a_\dagger)\times\omega)$ such that the solution p of (4) belongs to the orthogonal of E_μ at time T, for every $a\in(0,a_\dagger)$. The control cost behaves like $e^{C\sqrt{\mu}}$, and may be compensated by the natural dissipation of the solution - under assumption **(Stable)** - stated in the following Proposition.

Proposition 4. *Under the assumption* **(Stable)**, *let* $\mu>0$ *and suppose that* $\Pi_{E_\mu} p_0(a,\cdot) = 0$ *for almost every* $a\in(0,a_\dagger)$. *Then there exists a constant* $C\geqslant 0$ *such that for every* $t>0$, *the solution* p *of* (4) *with* $u\equiv 0$ *satisfies*

$$\|p(t,\cdot,\cdot)\|_{L^\infty((0,a_\dagger);L^2(\Omega))} \leqslant Ce^{-\mu t}\|p_0\|_{L^\infty((0,a_\dagger);L^2(\Omega))} \qquad (t\geqslant 0).$$

Proof. Suppose that $\Pi_{E_\mu} p_0(a) = 0$ for almost every $a\in(0,a_\dagger)$. With $u\equiv 0$, the solution p of (4) satisfies

$$p(t,a,\cdot) = \sum_{j:\ \lambda_j>\mu} p^j(t,a)\varphi_j \ \text{ in } L^2(\Omega),\ \text{ a.e. in } (0,\tau)\times(0,a_\dagger),$$

where

$$\begin{cases} \partial_t p^j(t,a) + \partial_a p^j(t,a) + (\mu(a)+\lambda_j)p^j(t,a) = 0 & t>0,\ a\in(0,a_\dagger), \\ p^j(t,0) = \displaystyle\int_0^{a_\dagger} \beta(a)p^j(t,a)\,\mathrm{d}a, & t>0, \\ p^j(0,a) = p_0^j(a), & a\in(0,a_\dagger). \end{cases} \tag{34}$$

Let $\lambda_j > \mu$. It is easy to check that the solution p^j of (34) satisfies $p^j(t,a) = e^{-t\lambda_j}\mathbb{T}_t^{A_0}(p_0^j)(a)$, so that for almost every $(t,a)\in(0,\tau)\times(0,a_\dagger)$ we have

$$\begin{aligned} \|p(t,a,\cdot)\|^2_{L^2(\Omega)} &= \sum_{j:\ \lambda_j>\mu} |p^j(t,a)|^2 \\ &\leqslant e^{-2\mu t}\sum_{j=0}^{+\infty}(\mathbb{T}_t^{A_0}p_0^j(a))^2, \end{aligned}$$

and using inequality (20) we deduce from the above inequality that we have

$$\|p(t,a,\cdot)\|^2_{L^2(\Omega)} \leqslant Ce^{-2\mu t}\|p_0\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))},$$

so that the estimation of Proposition 4 holds. □

In order to prove Theorem 1.2, we will need the following known and useful result from the measure theory, whose proof may be found in Lions [26, p. 275].

Lemma 5.1. *Let $T>0$ and E be a Lebesgue measurable set with positive measure in $[0,T]$. For almost every $\tilde{t}\in E$, there exists a sequence of numbers $\{t_i\}_{i=1}^\infty$ in the interval $[0,T]$ such that*

$$t_1 < t_2 < \cdots < t_i < t_{i+1} < \cdots < \tilde{t}, \quad t_i \to \tilde{t} \text{ as } i\to\infty, \tag{35}$$

$$m(E\cap[t_i,t_{i+1}]) \geqslant \rho(t_{i+1}-t_i), \quad i=1,2,\dots, \tag{36}$$

$$\frac{t_{i+1}-t_i}{t_{i+2}-t_{i+1}} \leqslant C_0, \quad i=1,2,\dots, \tag{37}$$

where $\rho\leqslant 1$ and C_0 are two positive constants which are independant on i.

We now have all the ingredients to prove Theorem 1.2, following the ideas developed in Wang [36].

Proof of Theorem (1.2). With no claim of originality, we borrow some ideas from [36] (see also Lü [28]). Without loss of generality, we may assume that the assumption **(Stable)** is satisfied (see Remark 2). We may also assume that $C_1\geqslant 1$, where C_1 is the positive constant given in (3) (resp. in (36)). Let $C\geqslant 1$ be a fixed constant such that (21) and (4) hold. By Lemma 5.1, we can take a number $\tilde{t}\in E$ with $\tilde{t}<T$ and a sequence $\{t_N\}_{N=1}^\infty$ in the interval $(0,T)$ such that (35) - (37) hold for some positive number $\rho\leqslant 1$ and C_0, and

$$\tilde{t}-t_1 \leqslant \min\left(1,\frac{1}{|\omega|}\right).$$

Let us consider the following equation:

$$\begin{cases} \partial_t \tilde{p}(t,a,x) + \partial_a \tilde{p}(t,a,x) + \mu(a)\tilde{p}(t,a,x) \\ \quad - \Delta \tilde{p}(t,a,x) = \chi_\omega(x)\chi_E(t)\tilde{u}(t,a,x), & t \in [t_1,\tilde{t}],\ a \in (0,a_\dagger),\ x \in \Omega, \\ \dfrac{\partial \tilde{p}}{\partial \nu}(t,a,x) = 0, & t \in [t_1,\tilde{t}],\ a \in (0,a_\dagger),\ x \in \partial\Omega, \\ \tilde{p}(t,0,x) = \displaystyle\int_0^{a_\dagger} \beta(a)\tilde{p}(t,a,x)\,\mathrm{d}a, & t \in [t_1,\tilde{t}],\ x \in \Omega, \\ \tilde{p}(t_1,a,x) = \widetilde{p_0}(a,x), & a \in (0,a_\dagger),\ x \in \Omega. \end{cases} \tag{38}$$

We shall first prove that for each $\widetilde{p_0} \in L^\infty((0,a_\dagger);L^2(\Omega))$, there exists a control $\tilde{u}$ in the space $L^\infty((t_1,\tilde{t}) \times (0,a_\dagger) \times \omega)$ with the estimate $\|\tilde{u}\|_{L^\infty((t_1,\tilde{t})\times(0,a_\dagger)\times\omega)} \leqslant L\|\widetilde{p_0}\|_{L^\infty((0,a_\dagger);L^2(\Omega))}$ for some positive constant L independant of $\widetilde{p_0}$, such that the solution $\tilde{p}$ to (38) vanishes at time $\tilde{t}$, *i.e.* $\tilde{p}(\tilde{t},a,x) = 0$ for almost every $(a,x) \in (0,a_\dagger) \times \Omega$.

Set $I_N := [t_{2N-1}, t_{2N}]$, $J_N := [t_{2N}, t_{2N+1}]$ for $N = 1,2,\ldots$. Then we have

$$[t_1,\tilde{t}) = \bigcup_{N=1}^{\infty} (I_N \cup J_N).$$

Notice that for each $N \geqslant 1$, we have $m(E \cap I_N) > 0$ thanks to (36).

Now, on the interval $I_1 \equiv [t_1,t_2]$, we consider the following controlled equation:

$$\begin{cases} \partial_t p_1(t,a,x) + \partial_a p_1(t,a,x) + \mu(a)p_1(t,a,x) \\ \quad - \Delta p_1(t,a,x) = \chi_\omega(x)\chi_E(t)u_1(t,a,x), & t \in [t_1,t_2],\ a \in (0,a_\dagger),\ x \in \Omega, \\ \dfrac{\partial p_1}{\partial \nu}(t,a,x) = 0, & t \in [t_1,t_2],\ a \in (0,a_\dagger),\ x \in \partial\Omega, \\ p_1(t,0,x) = \displaystyle\int_0^{a_\dagger} \beta(a)p_1(t,a,x)\,\mathrm{d}a, & t \in [t_1,t_2],\ x \in \Omega, \\ p_1(t_1,a,x) = \widetilde{p_0}(a,x), & a \in (0,a_\dagger),\ x \in \Omega. \end{cases} \tag{39}$$

By Proposition 3, for any $r_1 > 0$, there exists a control u_1 in the space $L^\infty((t_1,t_2) \times (0,a_\dagger) \times \omega)$ with the estimate:

$$\|u_1\|^2_{L^\infty((t_1,t_2)\times(0,a_\dagger)\times\omega)} \leqslant \frac{C_1 e^{C_2\sqrt{r_1}}}{(m(E\cap[t_1,t_2]))^2}\|\widetilde{p_0}\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))}, \tag{40}$$

such that $\Pi_{E_{r_1}} p_1(t_2,a,\cdot) = 0$ for almost every $a \in (0,a_\dagger)$. Then, using (36) with (40) we have

$$\begin{aligned} \|u_1\|^2_{L^\infty((t_1,t_2)\times(0,a_\dagger)\times\omega)} &\leqslant \frac{C_1 e^{C_2\sqrt{r_1}}}{\rho^2(t_2-t_1)^2}\|\widetilde{p_0}\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))} \\ &= \frac{C_1}{\rho^2(t_2-t_1)^2}\alpha_1\|\widetilde{p_0}\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))}, \end{aligned} \tag{41}$$

where $\alpha_1 := e^{C_2\sqrt{r_1}}$. Moreover, from (17), (18) and (21) we get that the solution p_1 of (39) satisfies

$$\|p_1(t_2,\cdot,\cdot)\|_{L^\infty((0,a_\dagger);L^2(\Omega))} \leqslant C\|\widetilde{p_0}\|_{L^\infty((0,a_\dagger);L^2(\Omega))} + C\int_{t_1}^{t_2} \|u_1(\sigma)\|_{L^\infty((0,a_\dagger);L^2(\omega))}\,\mathrm{d}\sigma$$

$$\begin{aligned}
&\leqslant C\|\widetilde{p_0}\|_{L^\infty((0,a_\dagger);L^2(\Omega))} + C(t_2-t_1)|\omega|\|u_1\|_{L^\infty((t_1,t_2)\times(0,a_\dagger)\times\omega)} \\
&\leqslant C(\|\widetilde{p_0}\|_{L^\infty((0,a_\dagger);L^2(\Omega))} + \|u_1\|_{L^\infty((t_1,t_2)\times(0,a_\dagger)\times\omega)}) \\
&\leqslant 2C\left(\frac{C_1}{\rho^2(t_2-t_1)^2}\alpha_1\right)^{\frac{1}{2}} \|\widetilde{p_0}\|_{L^\infty((0,a_\dagger);L^2(\Omega))}.
\end{aligned} \tag{42}$$

here, we have used the fact that $(t_2-t_1)\leqslant \min\left(1,\frac{1}{|\omega|}\right)$, $\rho\leqslant 1$ and $C_1\geqslant 1$, together with (41).

On the interval $J_1\equiv[t_2,t_3]$, we consider the following population equation without control:

$$\begin{cases}
\partial_t q_1(t,a,x)+\partial_a q_1(t,a,x)+\mu(a)q_1(t,a,x) & \\
\quad -\Delta p_1(t,a,x)=0, & t\in[t_2,t_3],\ a\in(0,a_\dagger),\ x\in\Omega,\\
\dfrac{\partial q_1}{\partial\nu}(t,a,x)=0, & t\in[t_2,t_3],\ a\in(0,a_\dagger),\ x\in\partial\Omega,\\
q_1(t,0,x)=\displaystyle\int_0^{a_\dagger}\beta(a)q_1(t,a,x)\,\mathrm{d}a, & t\in[t_2,t_3],\ x\in\Omega,\\
q_1(t_2,a,x)=p_1(t_2,a,x), & a\in(0,a_\dagger),\ x\in\Omega.
\end{cases}$$

Since $\Pi_{E_{r_1}}p_1(t_2,a,\cdot)=0$ for almost every $a\in(0,a_\dagger)$, from Proposition 4 we have

$$\|q_1(t_3,\cdot,\cdot)\|\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))}\leqslant C^2\exp(-2r_1(t_3-t_2))\cdot\|p_1(t_2,\cdot,\cdot)\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))}, \tag{43}$$

and using (42) with (43) we get that

$$\|q_1(t_3,\cdot,\cdot)\|\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))}\leqslant 4C^4\times\frac{C_1}{\rho^2(t_2-t_1)^2}\alpha_1\cdot\exp(-2r_1(t_3-t_2))\cdot\|\widetilde{p_0}\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))}. \tag{44}$$

On the interval $I_2\equiv[t_3,t_4]$, we consider the controlled population equation as follows:

$$\begin{cases}
\partial_t p_2(t,a,x)+\partial_a p_2(t,a,x)+\mu(a)p_2(t,a,x) & \\
\quad -\Delta p_2(t,a,x)=\chi_\omega(x)\chi_E(t)u_2(t,a,x), & t\in[t_3,t_4],\ a\in(0,a_\dagger),\ x\in\Omega,\\
\dfrac{\partial p_2}{\partial\nu}(t,a,x)=0, & t\in[t_3,t_4],\ a\in(0,a_\dagger),\ x\in\partial\Omega,\\
p_2(t,0,x)=\displaystyle\int_0^{a_\dagger}\beta(a)p_2(t,a,x)\,\mathrm{d}a, & t\in[t_3,t_4],\ x\in\Omega,\\
p_2(t_3,a,x)=q_1(t_3,a,x), & a\in(0,a_\dagger),\ x\in\Omega.
\end{cases} \tag{45}$$

Then by using Proposition 3, for any $r_2>0$, there exists a control u_2 in the space $L^\infty((t_3,t_4)\times(0,a_\dagger)\times\omega)$ with the estimate:

$$\|u_2\|^2_{L^\infty((t_3,t_4)\times(0,a_\dagger)\times\omega)}\leqslant\frac{C_1e^{C_2\sqrt{r_2}}}{(m(E\cap[t_3,t_4]))^2}\|q_1(t_3,\cdot,\cdot)\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))}, \tag{46}$$

such that $\Pi_{E_{r_2}}p_2(t_4,a,\cdot)=0$ for almost every $a\in(0,a_\dagger)$. By using (37), (44) and (46) we have

$$\|u_2\|^2_{L^\infty((t_3,t_4)\times(0,a_\dagger)\times\omega)}\leqslant 4C^4\left(\frac{C_1}{\rho^2(t_2-t_1)^2}\right)^2 C_0^4\cdot\alpha_1\cdot\alpha_2\cdot\|\widetilde{p_0}\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))}, \tag{47}$$

where $\alpha_2 := \exp(C_2\sqrt{r_2})\exp(-2r_1(t_3-t_2))$. Given (17), (18) and (21), we may infer from (47) that the solution p_2 of (45) satisfies

$$\|p_2(t_4,\cdot,\cdot)\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))} \leqslant 4^2C^6\left(\frac{C_1e^{C_2\sqrt{r_2}}}{\rho^2(t_2-t_1)^2}\right)^2 C_0^4\cdot\alpha_1\cdot\alpha_2\cdot\|\widetilde{p_0}\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))}. \tag{48}$$

On the interval $J_2 \equiv [t_4,t_5]$, we consider the following controlled population without control:

$$\begin{cases}\partial_t q_2(t,a,x)+\partial_a q_2(t,a,x)+\mu(a)q_2(t,a,x) & \\ \quad -\Delta q_2(t,a,x)=0, & t\in[t_4,t_5],\ a\in(0,a_\dagger),\ x\in\Omega,\\ \dfrac{\partial q_2}{\partial\nu}(t,a,x)=0, & t\in[t_4,t_5],\ a\in(0,a_\dagger),\ x\in\partial\Omega,\\ q_2(t,0,x)=\displaystyle\int_0^{a_\dagger}\beta(a)q_2(t,a,x)\,\mathrm{d}a, & t\in[t_4,t_5],\ x\in\Omega,\\ q_2(t_4,a,x)=p_2(t_4,a,x), & a\in(0,a_\dagger),\ x\in\Omega.\end{cases}$$

Since $\Pi_{E_{r_2}}p_2(t_4,a,\cdot)=0$ for almost every $a\in(0,a_\dagger)$, from Proposition (4) we have

$$\|q_2(t_5,\cdot,\cdot)\|\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))} \leqslant C^2\exp(-2r_2(t_5-t_4))\cdot\|p_2(t_4,\cdot,\cdot)\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))}, \tag{49}$$

and using (48) with (49) we get that

$$\begin{aligned}&\|q_2(t_5,\cdot,\cdot)\|\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))}\\ &\quad\leqslant 4^2C^8\left(\frac{C_1}{\rho^2(t_2-t_1)^2}\right)^2 C_0^4\cdot\alpha_1\cdot\alpha_2\cdot\exp(-2r_2(t_5-t_4))\cdot\|\widetilde{p_0}\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))}.\end{aligned} \tag{50}$$

On the interval $I_3 \equiv [t_5,t_6]$, we consider the controlled population equation as follows:

$$\begin{cases}\partial_t p_3(t,a,x)+\partial_a p_3(t,a,x)+\mu(a)p_3(t,a,x) & \\ \quad -\Delta p_3(t,a,x)=\chi_\omega(x)\chi_E(t)u_3(t,a,x), & t\in[t_5,t_6],\ a\in(0,a_\dagger),\ x\in\Omega,\\ \dfrac{\partial p_3}{\partial\nu}(t,a,x)=0, & t\in[t_5,t_6],\ a\in(0,a_\dagger),\ x\in\partial\Omega,\\ p_3(t,0,x)=\displaystyle\int_0^{a_\dagger}\beta(a)p_3(t,a,x)\,\mathrm{d}a, & t\in[t_5,t_6],\ x\in\Omega,\\ p_3(t_5,a,x)=q_2(t_5,a,x), & a\in(0,a_\dagger),\ x\in\Omega.\end{cases}$$

Then by using Proposition 3, for any $r_3>0$, there exists a control u_3 in the space $L^\infty((t_5,t_6)\times(0,a_\dagger)\times\omega)$ with the estimate:

$$\|u_3\|^2_{L^\infty((t_5,t_6)\times(0,a_\dagger)\times\omega)} \leqslant \frac{C_1e^{C_2\sqrt{r_3}}}{(m(E\cap[t_5,t_6]))^2}\|q_2(t_5,\cdot,\cdot)\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))}, \tag{51}$$

such that $\Pi_{E_{r_3}}p_3(t_6,a,\cdot)=0$ for almost every $a\in(0,a_\dagger)$. By using (37), (50) and (51) we have

$$\|u_3\|^2_{L^\infty((t_5,t_6)\times(0,a_\dagger)\times\omega)} \leqslant 4^2C^8\left(\frac{C_1}{\rho^2(t_2-t_1)^2}\right)^3 C_0^4C_0^{4\cdot 2}\cdot\alpha_1\cdot\alpha_2\cdot\alpha_3\|\widetilde{p_0}\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))},$$

where $\alpha_3 := \exp(C_2\sqrt{r_3})\exp(-2r_2(t_3-t_2)C_0^{-2})$.

Generally, on the interval I_N, we consider the following controlled population equation:

$$\begin{cases}\partial_t p_N(t,a,x)+\partial_a p_N(t,a,x)+\mu(a)p_N(t,a,x) & \\ \quad -\Delta p_N(t,a,x)=\chi_\omega(x)\chi_E(t)u_N(t,a,x), & t\in[t_{2N-1},t_{2N}],\ a\in(0,a_\dagger),\ x\in\Omega,\\ \dfrac{\partial p_N}{\partial\nu}(t,a,x)=0, & t\in[t_{2N-1},t_{2N}],\ a\in(0,a_\dagger),\ x\in\partial\Omega,\\ p_N(t,0,x)=\displaystyle\int_0^{a_\dagger}\beta(a)p_N(t,a,x)\,\mathrm{d}a, & t\in[t_{2N-1},t_{2N}],\ x\in\Omega,\\ p_N(t_{2N-1},a,x)=q_{N-1}(t_{2N-1},a,x), & a\in(0,a_\dagger),\ x\in\Omega.\end{cases} \tag{52}$$

On the interval J_N, we consider the following uncontrolled population equation:

$$\begin{cases}\partial_t q_N(t,a,x)+\partial_a q_N(t,a,x)+\mu(a)q_N(t,a,x) & \\ \quad -\Delta q_N(t,a,x)=0, & t\in[t_{2N},t_{2N+1}],\ a\in(0,a_\dagger),\ x\in\Omega,\\ \dfrac{\partial q_N}{\partial\nu}(t,a,x)=0, & t\in[t_{2N},t_{2N+1}],\ a\in(0,a_\dagger),\ x\in\partial\Omega,\\ q_N(t,0,x)=\displaystyle\int_0^{a_\dagger}\beta(a)q_N(t,a,x)\,\mathrm{d}a, & t\in[t_{2N},t_{2N+1}],\ x\in\Omega,\\ q_N(t_{2N},a,x)=p_N(t_{2N},a,x), & a\in(0,a_\dagger),\ x\in\Omega.\end{cases}$$

It may be shown by induction that, for each $r_N>0$, there exists a control $u_N\in L^\infty(I_N\times(0,a_\dagger)\times\omega)$ satisfying:

$$\begin{aligned}&\|u_N\|^2_{L^\infty(I_N\times(0,a_\dagger)\times\omega)}\\ &\leqslant (4C^4)^{N-1}\left(\frac{C_1}{\rho^2(t_2-t_1)^2}\right)^N C_0^4C_0^{4\times 2}\cdots C_0^{4(N-1)}\alpha_1\alpha_2\cdots\alpha_N\|\widetilde{p_0}\|^2_{L^\infty((0,a_\dagger);L^2(\Omega))},\end{aligned} \tag{53}$$

where

$$\alpha_N:=\begin{cases}\exp(C_2\sqrt{r_1}), & N=1,\\ \exp(C_2\sqrt{r_N})\exp(-2r_{N_1}(t_3-t_2)C_0^{-2(N-2)}), & N\geqslant 2,\end{cases}$$

such that $\Pi_{E_{r_N}}p_N(t_{2N},a,\cdot)=0$ for almost every $a\in(0,a_\dagger)$. It is easy to check that for each $N\geqslant 1$ we have

$$\|u_N\|_{L^\infty(I_N\times(0,a_\dagger)\times\omega)}\leqslant(\tilde{C})^{N(N-1)}\alpha_1\cdots\alpha_N\cdot\|\tilde{p_0}\|_{L^\infty((0,a_\dagger);L^2(\Omega))},$$

where

$$\tilde{C}:=\frac{4C^4C_1}{\rho^2(t_2-t_1)^2}\cdot C_0^2.$$

Now, for every $N\geqslant 1$ we set

$$r_N:=\left[\frac{2}{(t_3-t_2)}\tilde{C}^{N-1}\right]^4\qquad (N\geqslant 1).$$

With this choice of the sequence $\{r_N\}_{N\geqslant 1}$ given by the above formula, it is shown in [36] that the sequence $\{(\tilde{C})^{N(N-1)}\alpha_1\cdots\alpha_N\}_{N\geqslant 1}$ is bounded by some nonnegative constant L, so that from (53) it follows that for every $N\geqslant 1$ we have

$$\|u_N\|_{L^\infty(I_N\times(0,a_\dagger)\times\omega)}\leqslant L\|\tilde{p_0}\|_{L^\infty((0,a_\dagger);L^2(\Omega))}\qquad (N\geqslant 1). \tag{54}$$

We now construct a control $\tilde{u}$ by setting

$$\tilde{u}(t,a,x)=\begin{cases}u_N(t,a,x), & t\in I_N,\ a\in(0,a_\dagger),\ N\geqslant 1,\\ 0, & t\in J_N,\ a\in(0,a_\dagger),\ N\geqslant 1,\end{cases} \tag{55}$$

so that from (54) it is clear that $\tilde{u} \in L^\infty((t_1, \tilde{t}) \times (0, a_\dagger) \times \omega)$ with the estimate

$$\|\tilde{u}\|_{L^\infty((t_1,\tilde{t})\times(0,a_\dagger)\times\omega)} \leqslant L\|\tilde{p_0}\|_{L^\infty((0,a_\dagger);L^2(\Omega))}. \tag{56}$$

Denote by $\tilde{p}$ the solution of (38) corresponding to the control $\tilde{u}$ defined by (55). Then on any interval I_N, we have $\tilde{p}(t,\cdot,\cdot) = p_N(t,\cdot,\cdot)$, where p_N is the solution of (52). Since we have $\Pi_{E_{r_N}} p_N(t_{2N}, a, \cdot) = 0$ for almost every $a \in (0, a_\dagger)$ and for every $N \geqslant 1$, using the fact that $r_1 < r_2 < \cdots < r_N < \ldots$, by making use of (55) we get that

$$\Pi_{E_{r_N}} p_N(t_{2M}, a, \cdot) = 0 \quad \text{for all} \quad M \geqslant N. \tag{57}$$

On the other hand, since $t_{2M} \to \tilde{t}$ as $M \to \infty$, we obtain that

$$\tilde{p}(t_{2M},\cdot,\cdot) \to \tilde{p}(\tilde{t},\cdot,\cdot) \;\text{ strongly in } L^2((0,a_\dagger)\times\Omega) \text{ as } M \to \infty.$$

This, together with (57), implies that $\Pi_{E_{r_N}} p_N(\tilde{t}, a, \cdot) = 0$ for almost every $a \in (0, a_\dagger)$ and for every $N \geqslant 1$. Since $r_N \to \infty$ when $N \to \infty$, it holds that $\tilde{p}(\tilde{t},\cdot,\cdot) = 0$. Thus, we have proved that for each $\tilde{p_0} \in L^\infty((0, a_\dagger); L^2(\Omega))$, there exists a control $\tilde{u} \in L^\infty((t_1, \tilde{t}) \times (0, a_\dagger) \times \omega)$ with the estimate $\|\tilde{u}\|_{L^\infty((t_1,\tilde{t})\times(0,a_\dagger)\times\omega)} \leqslant L\|\tilde{p_0}\|_{L^\infty((0,a_\dagger);L^2(\Omega))}$ such that the solution $\tilde{p}$ to (38) reaches zero value at time $\tilde{t}$, namely, $\tilde{p}(\tilde{t},\cdot,\cdot) = 0$.

Now, we take $\tilde{p_0}$ to be $\mathbb{T}^A_{t_1} p_0$ and we construct a control u by setting, for almost every $(t, a, x) \in (0, \tau) \times (0, a_\dagger) \times \Omega$,

$$u(t,a,x) = \begin{cases} 0 & \text{in } (0,t_1)\times(0,a_\dagger)\times\Omega, \\ \tilde{u}(t,a,x) & \text{in } (t_1,\tilde{t})\times(0,a_\dagger)\times\Omega, \\ 0 & \text{in } (\tilde{t},\tau)\times(0,a_\dagger)\times\Omega. \end{cases} \tag{58}$$

It is clear that this control u is in the space $L^\infty((0, \tau) \times (0, a_\dagger) \times \Omega)$ and that the corresponding solution p of (4) satisfies $p(\tau,\cdot,\cdot) = 0$. The norm estimation (5) in Theorem 1.2 easily follows from (21), (56) and (58). □

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