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Null controllability via comparison results for nonlinear age-structured population dynamics

Nicolas Hegoburu¹ · Sebastian Aniţa^{2,3}

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Abstract

We consider an infinite dimensional nonlinear controlled system describing agestructured population dynamics, where the birth and the mortality rates are nonlinear functions of the population size. The control being active on some age range, we give sharp conditions subject to the age range and the control time horizon to get the null controllability of the nonlinear controlled population dynamics. The main novelty is that we use here as a main ingredient the comparison principle for age-structured population dynamics, and in case of null controllability we provide a feedback control with a very simple structure, while preserving the nonnegativity of the state trajectory. Finally, we establish the lack of the null controllability for the linear Lotka-McKendrick equation with spatial diffusion when the control acts in a subset of the habitat and we want to preserve the positivity of the state trajectory.

Keywords Population dynamics \cdot Null controllability \cdot Feedback controls \cdot Nonlinearities \cdot Nonnegativity

1 Introduction

We shall continue and extend the investigation in Hegoburu et al. [8] concerning the null controllability of the age-dependent population dynamics. More precisely, let p(t, a) be the distribution of individuals of age a at time t of a biological population. Let a_{\dagger} be the maximal age of an individual and τ be a positive constant. Consider

sanita@uaic.ro

Nicolas Hegoburu nicolas.hegoburu@math.u-bordeaux.fr
 Sebastian Aniţa

¹ Institut de Mathématiques de Bordeaux, Université de Bordeaux/Bordeaux INP/CNRS, 351 Cours de la Libération, 33 405 Talence, France

² Faculty of Mathematics, "Alexandru Ioan Cuza" University of Iaşi, Iaşi, Romania

³ "Octav Mayer" Institute of Mathematics of the Romanian Academy, 700506 Iaşi, Romania

that the population dynamics is described by the following nonlinear system, firstly proposed (without control) by Gurtin and MacCamy [7]:

$$\begin{aligned} \frac{\partial p}{\partial t}(t,a) &+ \frac{\partial p}{\partial a}(t,a) + \mu(t,a,P(t))p(t,a) = \chi_{[a_1,a_2]}(a)u(t,a), & (t,a) \in Q_{\tau} \\ p(t,0) &= \int_{0}^{a_{\dagger}} \beta(t,a,P(t))p(t,a) \, \mathrm{d}a, & t \in (0,\tau) \\ p(0,a) &= p_0(a), & a \in (0,a_{\dagger}) \end{aligned}$$
(1)
$$P(t) &= \int_{0}^{a_{\dagger}} p(t,a) \, \mathrm{d}a, & t \in (0,\tau), \end{aligned}$$

where *u* is a control function, $\chi_{[a_1,a_2]}$ is the characteristic function of the interval $[a_1, a_2]$ (where $0 \leq a_1 < a_2 \leq a_{\dagger}$), p_0 is the initial population density and $Q_{\tau} = (0, \tau) \times (0, a_{\dagger})$. β and μ are the fertility rate and the mortality rate, respectively. In the following, the nonlocal boundary condition

$$p(t,0) = \int_0^{a_{\dagger}} \beta(t, a, P(t)) p(t, a) \, \mathrm{d}a, \quad t \in (0, \tau),$$

will be referred as the renewal law.

To state our main results, we first recall the standard assumptions, used for instance in Aniţa [5, p. 30], on the functions μ , β , p_0 and u:

- (H1) for every $s \ge 0$, the functions $\beta(\cdot, \cdot, s)$ and $\mu(\cdot, \cdot, s)$ belong to $L^{\infty}(Q_{\tau})$ and $L^{1}_{loc}([0, \tau] \times [0, a_{\dagger}))$ respectively,
- (H2) the functions β and μ are locally Lipschitz functions with respect to the third variable, i.e., for any M > 0, there exists L(M) > 0 such that for almost every $(t, a) \in Q_{\tau}$ and for every $s_1, s_2 \in [0, M]$, we have

$$|\beta(t, a, s_1) - \beta(t, a, s_2)| \leq L(M) \cdot |s_1 - s_2|,$$

$$|\mu(t, a, s_1) - \mu(t, a, s_2)| \leq L(M) \cdot |s_1 - s_2|,$$

(H3) for almost every $(t, a, s) \in Q_{\tau} \times (0, +\infty)$, we have

$$\beta(t, a, s) \ge 0$$
 and $\mu(t, a, s) \ge 0$,

- (H4) for almost every $(t, a) \in Q_{\tau}$, the function $\beta(t, a, \cdot)$ is nonincreasing,
- **(H5)** for almost every $(t, a) \in Q_{\tau}$, the function $\mu(t, a, \cdot)$ is nondecreasing,
- (**H6**) $p_0 \in L^2(0, a_{\dagger}), p_0(a) \ge 0$ a.e. $a \in (0, a_{\dagger}),$
- (H7) the control function u belongs to $L^2(Q_{\tau})$.

By a solution to (1), we mean a function $p \in L^{\infty}(0, \tau; L^2(0, a_{\dagger}))$, absolutely continuous along almost every characteristic line of equation t - a = c [with $c \in (-a_{\dagger}, \tau)$], which satisfies

$$\begin{cases} Dp(t, a) + \mu(t, a, P(t))p(t, a) = \chi_{[a_1, a_2]}(a)u(t, a) & \text{a.e. } (t, a) \in Q_{\tau} \\ \lim_{\varepsilon \to 0^+} p(t + \varepsilon, \varepsilon) = \int_0^{a_{\dagger}} \beta(t, a, P(t))p(t, a)da & \text{a.e. } t \in (0, \tau) \\ \lim_{\varepsilon \to 0^+} p(\varepsilon, a + \varepsilon) = p_0(a) & \text{a.e. } a \in (0, a_{\dagger}), \end{cases}$$

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where $P(t) = \int_0^{a_{\dagger}} p(t, a) da$ for almost every $t \in (0, \tau)$. Here

$$Dp(t, a) = \lim_{\varepsilon \to 0} \frac{p(t + \varepsilon, a + \varepsilon) - p(t, a)}{\varepsilon}$$

is the directional derivative of p at (t, a) for direction (1, 1). The definition of the solution implies that there is a trace of p on any curve of equation t = c, with $c \in [0, \tau]$. With the above assumptions, system (1) has at most one nonnegative solution, and if in addition $u(t, a) \ge 0$ a.e. $(t, a) \in Q_{\tau}$, the existence and uniqueness of a nonnegative solution to (1) can be proved via the Banach fixed point theorem (see, for instance, [5, Chapter 2]).

Notice that actually the control acts only for $a \in [a_1, a_2]$.

For almost every $(t, a) \in Q_{\tau}$, denote by $\beta_+(t, a) := \beta(t, a, 0)$ so that, due to assumption **(H4)**, for almost every $(t, a, s) \in Q_{\tau} \times (0, +\infty)$ we have

$$\beta(t, a, s) \leq \beta(t, a, 0) = \beta_{+}(t, a)$$
 a.e. $(t, a, s) \in Q_{\tau} \times (0, +\infty)$. (2)

Our first result asserts that, assuming that the age of individuals able to reproduce is bounded away from zero, given a_1 small enough and τ large enough, we are able to bring the solution p of (1) to zero by means of a control function $u \in L^2(Q_\tau)$, preserving the nonnegativity of the state trajectory. More precisely, we have

Theorem 1 With the above notations and assumptions, suppose that there exists $a_b \in (0, a_{\dagger})$ such that for almost every $t \in (0, \tau)$, we have

$$\beta_{+}(t,a) = 0 \quad a.e. \ a \in (0, a_b), \tag{3}$$

where the function β_+ is defined in (2).

Assume that we have $a_1 < a_b$. If $\tau > a_{\dagger} - a_2 + a_1$, then for every $p_0 \in L^2(0, a_{\dagger})$ with $p_0(a) \ge 0$ a.e. $a \in (0, a_{\dagger})$, there exists $u \in L^2(Q_{\tau})$ and a corresponding unique nonnegative solution p of (1) such that

$$p(\tau, a) = 0$$
 a.e. $a \in (0, a_{\dagger})$.

The following Theorem 2 states that the previous result is sharp, in the sense that if a_1 (respectively τ) is bigger than a_b (respectively smaller than $a_{\dagger} - a_2 + a_1$), the system (1) may not be null controllable by means of controls $u \in L^2(Q_{\tau})$ while preserving the nonnegativity of the controlled state trajectory. More precisely, we have

Theorem 2 With the above notations and assumptions,

(i) Assume that there exists a_b ∈ (0, a_†) and a_m ∈ (0, a_†] with a_b < a_m such that, for almost every t ∈ (0, τ) we have

$$\beta(t, a, s) > 0$$
 a.e. $(a, s) \in (a_b, a_m) \times (0, +\infty)$.

If $a_1 > a_b$, then for every $\tau > 0$, there exists $p_0 \in L^2(0, a_{\dagger})$ with $p_0(a) \ge 0$ a.e. $a \in (0, a_{\dagger})$ such that there is no control u and a corresponding nonnegative solution p to (1) satisfying

$$p(\tau, a) = 0 \ a.e. \ a \in (0, a_{\dagger}).$$

(ii) If $\tau < a_{\dagger} - a_2 + a_1$, there exists a function β satisfying (H1)–(H4) and an initial population density $p_0 \in L^2(0, a_{\dagger})$ with $p_0(a) \ge 0$ a.e. $a \in (0, a_{\dagger})$ such that there is no control u and a corresponding nonnegative solution p to (1) satisfying

$$p(\tau, a) = 0$$
 a.e. $a \in (0, a_{\dagger})$.

Our third result asserts that condition (3) is not necessary to get the null controllability of system (1), provided that we are able to control the very young individuals. More precisely, provided that $a_1 = 0$ and $a_2 > 0$, for any function β satisfying (H1)–(H4), we are able to bring the solution p of (1) to zero in sufficiently large time τ by means of a control $u \in L^2(Q_{\tau})$, preserving the nonnegativity of the state trajectory:

Theorem 3 With the above notations and assumptions,

(i) Assume that $a_1 = 0$ and $a_2 > 0$. If $\tau > a_{\dagger} - a_2$, then for every $p_0 \in L^2(0, a_{\dagger})$ with $p_0(a) \ge 0$ a.e. $a \in (0, a_{\dagger})$, there exists $u \in L^2(Q_{\tau})$ and a unique nonnegative solution p of (1) such that

$$p(\tau, a) = 0 \ a.e. \ a \in (0, a_{\dagger}).$$

(ii) Assume that there exists $a_m \in (0, a_{\dagger}]$ such that, for almost every $t \in (0, \tau)$ we have

$$\beta(t, a, s) > 0$$
 a.e. $(a, s) \in (0, a_m) \times (0, +\infty)$.

If $a_1 > 0$, then for every $\tau > 0$, there exists $p_0 \in L^2(0, a_{\dagger})$ with $p_0(a) \ge 0$ a.e. $a \in (0, a_{\dagger})$ such that there is no control u and a corresponding nonnegative solution p to (1) satisfying

$$p(\tau, a) = 0$$
 a.e. $a \in (0, a_{\dagger})$.

(iii) If $\tau < a_{\dagger} - a_2$, there exists an initial population density $p_0 \in L^2(0, a_{\dagger})$ with $p_0(a) \ge 0$ a.e. $a \in (0, a_{\dagger})$ such that there is no control u and a corresponding nonnegative solution p to (1) satisfying

$$p(\tau, a) = 0 \ a.e. \ a \in (0, a_{\dagger}).$$

For an overview on age-structured population dynamics models, we refer, for instance, to Webb [19], Iannelli [10], Kunisch et al. [13] and the references therein.

In the case where both functions β and μ depend only on the age variable, some null controllability results of the age-dependent population dynamics model (without

diffusion) were first obtained by Barbu et al. [6]. Assuming that the control is supported in the age interval $(0, a_0)$, for some $a_0 < a_{\dagger}$, the authors proved that the controlled population may be steered to any steady state of (1), except for a small interval of ages near zero. Recently, Hegoburu et al. [8] proved that this restriction is not necessary, provided individuals do not reproduce at the age close to zero. They also proved there exists controls which preserves the nonnegativity of the state trajectory. More recently, Maity [14] proved that null controllability can be achieved by controls supported in any subinterval $[a_1, a_2]$ of $[0, a_{\dagger}]$, provided we control before the individuals start to reproduce. In this case, the author proved that the system is null controllable in sufficiently large time, i.e., when $\tau > a_1 + \max\{a_1, a_{\dagger} - a_2\}$ by means of observability inequalities.

The null controllability of systems modeling nonlinear age-structured population dynamics has been studied by Ainseba et al. [3], in the case when $a_1 = 0$ and $a_2 \in$ $(0, a_{\dagger})$, using as a main ingredient the Kakutani fixed point theorem. In [3], the main result asserts controllability of the system (1) to zero in time $\tau > a_{\dagger}$ by a distributed control, except for a small interval of ages near zero. In our present paper, we get an exact null controllability result for the system (1). Moreover, the minimal time needed to steer the population to zero (i.e., $\tau > a_{\dagger} - a_2 + a_1$) seems to be sharp. Instead of using observability results, we use here as a main ingredient the comparison principle for linear age-structured population dynamics. In case of null controllability, we provide a feedback control with a very simple structure and such that (1) admits a unique nonnegative solution *p* satisfying $p(\tau, a) = 0$ a.e. $a \in (0, a_{\dagger})$.

As already mentioned, in the present work we use comparison results for agestructured population dynamics and some feedback controls with a simple structure (as a harvesting term) in order to prove some null controllability results. Actually, the use of such harvesting term in order to get the null controllability in an abstract space is presented in Sect. 2. Sections 3, 4 and 5 are devoted to prove, respectively, Theorems 1, 2 and 3, firstly in the case where the functions β and μ do not depend on the third variable (namely, in the *linear setting*) and to deduce the results of the abovementioned Theorems in the more general nonlinear setting by using the results of the linear case and the comparison principle for age-structured population dynamics. In Sect. 6, we give an application to the results obtained in the linear case: We show how to steer a population to another one in the linear setting, preserving the nonnegativity of the population. In Sect. 7, we show how the comparison principle for age-structured population dynamics may be used to prove the lack of the null controllability for the linear Lotka-McKendrick equation with spatial diffusion when the control acts in a subset of the habitat.

2 A null-controllability result in an abstract space

Consider the following controlled system:

$$\begin{cases} y'(t) - Ay(t) = u(t), & t \in (0, \tau) \\ y(0) = y_0, \end{cases}$$
(4)

where *A* is the generator of a C_0 -semigroup $(e^{tA})_{t\geq 0}$ of linear and bounded operators in the real Banach space *X* and τ is a positive constant. We remind that by a mild solution to (4), we mean the function $y \in C([0, \tau]; X)$ given by

$$y(t) = e^{tA}y_0 + \int_0^t e^{(t-s)A}u(s) \,\mathrm{d}s, \text{ for every } t \in [0, \tau].$$

If $y_0 \in X$ and $u \in L^2(0, \tau; X)$, then (4) has a unique mild solution.

The controllability properties of (4) have been investigated by Tucsnak and Weiss [18] via the observability. Here we use a different approach in order to get the null-controllability: we use a feedback control. Actually, if

$$u(t) := -\frac{1}{\tau - t} y(t), \quad t \in (0, \tau),$$
(5)

then (4) becomes

$$\begin{cases} y'(t) - Ay(t) = -\frac{1}{\tau - t}y(t), & t \in (0, \tau) \\ y(0) = y_0, \end{cases}$$
(6)

and by a mild solution to (6) we mean a function $y \in C([0, \tau]; X)$ such that

$$y(t) = e^{tA}y_0 - \int_0^t e^{(t-s)A}\left(\frac{1}{\tau-s}y(s)\right) ds$$
, for every $t \in [0, \tau]$.

By Gronwall–Bellman's inequality, we may conclude that there exists at most one mild solution to (6). Let us prove that actually, the function *y* defined by

$$y(t) := \frac{\tau - t}{\tau} e^{tA} y_0 \quad t \in [0, \tau],$$
(7)

is a mild solution to (6) and that u given by (5) belongs to $L^2(0, \tau; X)$.

Indeed, y given by (7) belongs to $C([0, \tau]; X)$ and for any $t \in [0, \tau]$ we have

$$e^{tA}y_0 - \int_0^t \frac{1}{\tau - s} e^{(t-s)A}y(s) ds$$

= $e^{tA}y_0 - \frac{1}{\tau} \int_0^t \frac{1}{\tau - s} (\tau - s)e^{(t-s)A}e^{sA}y_0 ds$
= $e^{tA}y_0 - \frac{t}{\tau}e^{tA}y_0 = \frac{\tau - t}{\tau}e^{tA}y_0 = y(t).$

On the other hand, the control u given by (5) satisfies

$$u(t) = -\frac{1}{\tau - t}y(t) = -\frac{1}{\tau}e^{tA}y_0, \quad t \in [0, \tau],$$

which indeed belongs to $C([0, \tau]; X)$ and consequently to $L^2(0, \tau; X)$.

Notice that, assuming that the functions β and μ depend only on the age variable, the formulation of Eq. (1) may be considered using semigroup theory. Indeed, denote by $X = L^2(0, a_{\dagger})$ and consider the operator $A : \mathcal{D}(A) \to X$ defined by

$$\mathcal{D}(A) = \left\{ \varphi \in L^2(0, a_{\dagger}); \ \varphi(0) = \int_0^{a_{\dagger}} \beta(a)\varphi(a) \, \mathrm{d}a, \ -\frac{\mathrm{d}\varphi}{\mathrm{d}a} - \mu\varphi \in L^2(0, a_{\dagger}) \right\},$$
$$A\varphi = -\frac{\mathrm{d}\varphi}{\mathrm{d}a} - \mu\varphi, \quad \varphi \in \mathcal{D}(A).$$

It is well known (see, for instance, Song et al. [16] or Kappel and Zhang [11]) that *A* generates a C_0 -semigroup of linear and bounded operators in *X* which we denote by $(e^{tA})_{t \ge 0}$. If the control function *u* in (1) acts in the whole age range $(0, a_{\dagger})$ (i.e., if $a_1 = 0$ and $a_2 = a_{\dagger}$) and is defined by the feedback form

$$u(t) := -\frac{1}{\tau - t} p(t), \quad t \in [0, \tau],$$
(8)

where p is the corresponding solution to (1), we deduce that p satisfies

$$\begin{cases} \frac{\partial p}{\partial t}(t,a) + \frac{\partial p}{\partial a}(t,a) + \left(\mu(a) + \frac{1}{\tau - t}\right) p(t,a) = 0, & (t,a) \in Q_{\tau} \\ p(t,0) = \int_{0}^{a_{\dagger}} \beta(a) p(t,a) \, \mathrm{d}a, & t \in (0,\tau) \\ p(0,a) = p_{0}(a), & a \in (0,a_{\dagger}), \end{cases}$$
(9)

and so that for every $t \in [0, \tau]$ we get from (7) that $p(t) = \frac{\tau - t}{\tau} e^{tA} p_0$ in *X* for any $t \in [0, \tau]$, and $p(\tau) = 0$ in *X*. It can be seen in (9) that, *u* given in such a feedback form (8), it behaves like an additional mortality (as a harvesting rate) for the age-structured population dynamics. In Sects. 3 and 5, we shall develop this idea in the more general case when the functions β and μ depend on both time and age variable, and the control *u* acts in a subinterval of $[0, a_{\dagger}]$.

3 A positive null controllability result when the young individuals do not reproduce

This section is devoted to prove Theorem 1, firstly in the case where the functions β and μ do not depend on the third variable (see Proposition 1). The proof of Theorem 1 in the more general nonlinear case will be deduced from the result of Proposition 1 and the comparison principle for age-structured population dynamics.

More precisely, we first assume that the population dynamics is described by the following linear system:

$$\frac{\partial p}{\partial t}(t,a) + \frac{\partial p}{\partial a}(t,a) + \mu_0(t,a)p(t,a) = \chi_{[a_1,a_2]}(a)u(t,a), \quad (t,a) \in Q_\tau$$

$$p(t,0) = \int_0^{a_{\dagger}} \beta_0(t,a)p(t,a) \, \mathrm{d}a, \qquad t \in (0,\tau) \quad (10)$$

$$p(0,a) = p_0(a), \qquad a \in (0,a_{\dagger}),$$

where τ is a positive constant and $Q_{\tau} = (0, \tau) \times (0, a_{\dagger})$. For the sake of completeness, let us recall the classical assumptions on the functions μ_0 , β_0 , p_0 and u relatively to the linear case:

(L1) the functions β_0 and μ_0 belong to $L^{\infty}(Q_{\tau})$ and $L^1_{loc}([0, \tau] \times [0, a_{\dagger}))$ respectively, (L2) for almost every $(t, a) \in Q_{\tau}$, we have

 $\beta_0(t, a) \ge 0$ and $\mu_0(t, a) \ge 0$,

(L3) $p_0 \in L^2(0, a_{\dagger}), p_0(a) \ge 0$ a.e. $a \in (0, a_{\dagger}),$ (L4) the control function *u* belongs to $L^2(Q_{\tau})$.

Note that, using the definition of a solution to (10), we can obtain, using an integration along the characteristic lines (see, for instance, [5, p. 16]) that for almost every $(t, a) \in Q_{\tau}$ with $a \ge t$, we have that the solution p to (10) satisfies

$$p(t, a) = \exp\left\{-\int_{0}^{t} \mu_{0}(s, a - t + s) \,\mathrm{d}s\right\} p_{0}(a - t)$$

$$+ \int_{0}^{t} \exp\left\{-\int_{s}^{t} \mu_{0}(\sigma, a - t + \sigma) \,\mathrm{d}\sigma\right\} \chi_{[a_{1}, a_{2}]}(a - t + s)u(s, a - t + s) \,\mathrm{d}s,$$
(11)

and for almost every $(t, a) \in Q_{\tau}$ with t > a, we have

$$p(t, a) = \exp\left\{-\int_{0}^{a} \mu_{0}(t - a + s, s) \,\mathrm{d}s\right\} B(t - a) + \int_{0}^{a} \exp\left\{-\int_{s}^{a} \mu_{0}(t - a + \sigma, \sigma) \,\mathrm{d}\sigma\right\} \chi_{[a_{1}, a_{2}]}(s)u(t - a + s, s) \,\mathrm{d}s,$$

where

$$B(t) = \int_0^{a_{\uparrow}} \beta_0(t, a) p(t, a) \,\mathrm{d}a \quad \text{a.e. } t \in (0, \tau).$$

Notice that, as a consequence of the Banach fixed point theorem, we get that $B \in L^{\infty}(0, \tau)$ (see, for instance, [5]).

In the following, we shall make several uses of the *comparison principle for age-structured population dynamics* (see [5, Theorem 2.1.2]) which we recall in the following Theorem 4.

(i) if $u(t, a) \ge 0$ a.e. in Q_{τ} , then $p(t, a) \ge 0$ a.e. in Q_{τ} ; (ii) if $\beta_{0i}, \mu_{0i}, p_{0i}, u_i$ satisfy (L1)–(L4) $(i = \overline{1, 2})$ and

$$\begin{aligned} \beta_{01}(t,a) &\geq \beta_{02}(t,a), \quad \mu_{01}(t,a) \leq \mu_{02}(t,a) \quad \text{a.e. in } Q_{\tau}, \\ p_{01}(a) &\geq p_{02}(a), \quad \text{a.e. in } (0,a_{\dagger}), \\ u_1(t,a) &\geq u_2(t,a), \quad \text{a.e. in } Q_{\tau}, \end{aligned}$$

then $p^1(t, a) \ge p^2(t, a)$ a.e. in Q_{τ} , where p^i is the solution of (10), corresponding to $\beta_0 := \beta_{0i}, \mu_0 := \mu_{0i}, p_0 := p_{0i}, u := u_i, i \in \overline{1, 2}$.

In this section, we will first prove the following Proposition 1 which is, roughly speaking, the "linear version" of Theorem 1.

Proposition 1 With the above notations and assumptions, suppose that there exists $a_b \in (0, a_{\dagger})$ such that for almost every $t \in (0, \tau)$, we have

$$\beta_0(t, a) = 0 \quad a.e. \ a \in (0, a_b).$$
(12)

Assume that we have $a_1 < a_b$. If $\tau > a_{\dagger} - a_2 + a_1$, then for every $p_0 \in L^2(0, a_{\dagger})$ with $p_0(a) \ge 0$ a.e. $a \in (0, a_{\dagger})$, there exists $u \in L^2(Q_{\tau})$ and a corresponding unique nonnegative solution p of (10) such that

$$p(\tau, a) = 0 \ a.e. \ a \in (0, a_{\dagger}).$$

In this whole section, we suppose that there exists $a_b > 0$ such that (12) is satisfied, with $a_1 < a_b$ and $\tau > a_{\dagger} - a_2 + a_1$.

We may divide the proof of Proposition 1 in two cases, since the shapes of the controls functions driving the initial population to zero in time τ may be slightly different in the two following cases, depending on the order relationship of a_2 and a_b :

Case 1:
$$\tau > a_{\dagger} - a_2 + a_1$$
 and $a_1 < a_2 \le a_b$,
Case 2: $\tau > a_{\dagger} - a_2 + a_1$ and $a_1 < a_b < a_2$.

The two following subsections are devoted to prove Proposition 1 in the two abovementioned cases.

3.1 The first case: $\tau > a_{\uparrow} - a_2 + a_1$ and $a_1 < a_2 \leq a_b$

This subsection is devoted to prove Proposition 1, in the case where $\tau > a_{\dagger} - a_2 + a_1$ and $a_1 < a_2 \leq a_b$.

Proof of Proposition 1 *with* $a_2 \leq a_b$ Suppose that $\tau > a_{\dagger} - a_2 + a_1$ and $a_1 < a_2 \leq a_b$. Without loss of generality, we may assume that $\tau < a_{\dagger}$. Let us prove that there exists a feedback control u and a corresponding unique nonnegative solution p of (10) such that

$$p(\tau, a) = 0$$
 a.e. $a \in (0, a_{\dagger})$.

Denote by $\varepsilon := \frac{1}{2}(\tau - (a_{\dagger} - a_2 + a_1))$. For almost every $(t, a) \in Q_{\tau}$, let us consider the feedback control u(t, a) := -v(t, a)p(t, a), where v is a control itself and acts as a harvesting rate, and is given by

$$v(t,a) := \begin{cases} \frac{1}{a_2 - a} & \text{if } \begin{cases} \varepsilon - a_2 \leqslant t - a < \tau - a_2, \\ a \in [a_1, a_2], \\ \tau - a_2 \leqslant t - a \leqslant \tau - (a_1 + \varepsilon), \\ a \in [a_1, a_2], \\ 0 & \text{otherwise.} \end{cases}$$
(13)

The structure of v is given in Fig. 1. The solution to (10) with u = -vp exists is unique and nonnegative (see Anita [5]), and via the comparison principle for age-dependent population dynamics [see Theorem 4 (ii)] we get that



$$0 \leqslant p(t,a) \leqslant \tilde{p}(t,a) \text{ a.e. } (t,a) \in Q_{\tau}, \tag{14}$$

Fig. 1 The structure of the control v when $\tau > a_{\dagger} - a_2 + a_1$ and $a_1 < a_2 \leq a_b$

where \tilde{p} is the solution to

$$\begin{cases} \frac{\partial \tilde{p}}{\partial t}(t,a) + \frac{\partial \tilde{p}}{\partial a}(t,a) + v(t,a)\tilde{p}(t,a) = 0, & (t,a) \in Q_{\tau} \\ \tilde{p}(t,0) = \int_{a_{b}}^{a_{\dagger}} \beta_{0}(t,a)\tilde{p}(t,a) \,\mathrm{d}a, & t \in (0,\tau) \\ \tilde{p}(0,a) = p_{0}(a), & a \in (0,a_{\dagger}). \end{cases}$$
(15)

Integrating along the characteristic lines, we get that the solution \tilde{p} of (15) is equal to

$$\begin{cases} p_0(a-t)\exp\left(-\int_{a-t}^{a}\frac{\mathrm{d}s}{a_2-s}\right) & \text{if } \begin{cases} \varepsilon - a_2 \leqslant t-a \leqslant -a_1, \\ a \in [a_1, a_2], \\ -a_1 \leqslant t-a \leqslant 0, \\ a \in [a_1, a_2], \end{cases} \\ \tilde{p}(t-a,0)\exp\left(-\int_{a_1}^{a}\frac{\mathrm{d}s}{a_2-s}\right) & \text{if } \\ \tilde{p}(t-a,0)\exp\left(-\int_{t-a+a_1}^{t}\frac{\mathrm{d}s}{\tau-s}\right) & \text{if } \begin{cases} 0 < t-a < \tau - a_2, \\ a \in [a_1, a_2], \\ 0 < t-a < \tau - a_2, \\ a \in [a_1, a_2], \end{cases} \\ \tau - a_2 \leqslant t-a \leqslant \tau - (a_1+\varepsilon), \\ a \in [a_1, a_2], \end{cases}$$

so that we have

$$\tilde{p}(t,a) = \begin{cases} \frac{a_2 - a}{a_2 - (a - t)} p_0(a - t) & \text{if} \\ \frac{a_2 - a}{a_2 - a_1} p_0(a - t) & \text{if} \\ \frac{a_2 - a}{a_2 - a_1} \tilde{p}(t - a, 0) & \text{if} \\ \frac{a_2 - a}{a_2 - a_1} \tilde{p}(t - a, 0) & \text{if} \\ \frac{\tau - t}{\tau - (t - a + a_1)} \tilde{p}(t - a, 0) & \text{if} \\ \end{cases} \begin{cases} \varepsilon - a_2 \leqslant t - a \leqslant -a_1, \\ a \in [a_1, a_2], \\ 0 < t - a < \tau - a_2, \\ a \in [a_1, a_2], \\ \tau - a_2 \leqslant t - a \leqslant \tau - (a_1 + \varepsilon), \\ a \in [a_1, a_2]. \end{cases}$$
(16)

We may conclude from the above formula that we have

$$\begin{cases} \tilde{p}(t, a_2) = 0 & \text{if } t \in (\varepsilon, \tau), \\ \tilde{p}(\tau, a) = 0 & \text{if } a \in (a_1 + \varepsilon, a_2). \end{cases}$$
(17)

Integrating on the characteristic lines the equation satisfied by \tilde{p} , we get from (17) that

$$\tilde{p}(t,a) = 0 \quad \text{if } (t,a) \in [\tau - (a_1 + \varepsilon), \tau] \times [a_2, a_{\dagger}]. \tag{18}$$

Since $a_b \ge a_2$, we obviously get from the above equation that

$$\tilde{p}(t,a) = 0$$
 if $(t,a) \in [\tau - (a_1 + \varepsilon), \tau] \times [a_b, a_{\dagger}].$

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Since \tilde{p} satisfies the renewal law, we may infer that

$$\tilde{p}(t,0) = 0$$
 if $t \in [\tau - (a_1 + \varepsilon), \tau]$,

and integrating along the characteristic lines we get that

$$\tilde{p}(\tau, a) = 0 \quad \text{if } a \in [0, a_1 + \varepsilon]. \tag{19}$$

By (17), (18) and (19), we may conclude that

$$\tilde{p}(\tau, a) = 0$$
 a.e. $a \in (0, a_{\dagger})$.

It remains to prove that the control u = -vp belongs to $L^2(Q_\tau)$. For almost every $(t, a) \in Q_\tau$, from (13) and (16) we have

$$v(t,a)\tilde{p}(t,a) = \begin{cases} \frac{p_0(a-t)}{a_2-(a-t)} & \text{if} \\ \frac{p_0(a-t)}{a_2-a_1} & \text{if} \\ \frac{\tilde{p}(t-a,0)}{a_2-a_1} & \text{if} \\ \frac{\tilde{p}(t-a,0)}{a_2-a_1} & \text{if} \\ \frac{\tilde{p}(t-a,0)}{a_2-a_1} & \text{if} \\ \frac{\tilde{p}(t-a,0)}{\tau-(t-a+a_1)} & \text{if} \\ \frac{\tau-a_2 \leqslant t-a \leqslant \tau-(a_1+\varepsilon)}{a \in [a_1,a_2],} \\ \tau-a_2 \leqslant t-a \leqslant \tau-(a_1+\varepsilon), \\ a \in [a_1,a_2]. \\ 0 & \text{otherwise,} \end{cases}$$

and this implies that

$$|v(t,a)\tilde{p}(t,a)| \leqslant \begin{cases} \frac{|p_{0}(a-t)|}{\varepsilon} & \text{if} \\ \frac{|p_{0}(a-t)|}{a_{2}-a_{1}} & \text{if} \\ \frac{|\tilde{p}(t-a,0)|}{a_{2}-a_{1}} & \text{if} \\ \frac{|\tilde{p}(t-a,0)|}{a_{2}-a_{1}} & \text{if} \\ \frac{|\tilde{p}(t-a,0)|}{\varepsilon} & \text{otherwise.} \end{cases}$$
(20)

Since $\tilde{p} \in L^2(Q_\tau)$, we may infer from the renewal law that we have $\tilde{p}(\cdot, 0) \in L^2(0, \tau)$, so that from (20) we have $v\tilde{p} \in L^2(Q_\tau)$. Given (14), it follows that we have $vp \in L^2(Q_\tau)$, which concludes the proof of Proposition 1 in the case where $a_2 \leq a_b$. \Box

3.2 The second case: $\tau > a_{\dagger} - a_2 + a_1$ and $a_1 < a_b < a_2$

This subsection is devoted to prove Proposition 1, in the case where $\tau > a_{\dagger} - a_2 + a_1$ and $a_1 < a_b < a_2$.

Proof of Proposition 1 with $a_b < a_2$ Suppose that $\tau > a_{\dagger} - a_2 + a_1$ and $a_1 < a_b < a_2$. Without loss of generality, we may assume that $\tau < a_{\dagger}$. We may also assume, without loss of generality, that we have $a_b \in (a_1, \tau)$: Indeed, in the case when there exists $a_b > 0$ satisfying (12) such that $a_b \ge \tau$, we may consider \tilde{a}_b such that $\tilde{a}_b \in (a_1, \tau)$ and replace a_b by \tilde{a}_b , noting that \tilde{a}_b also satisfies (12) since $\tilde{a}_b < a_b$.

For any $\varepsilon > 0$, denote by $\theta(\varepsilon) := \tau - (a_{\dagger} - a_2 + a_1) - \varepsilon$. Let us introduce $\varepsilon > 0$ small enough such that we have $a_1 + \varepsilon < a_b$ and $\theta(\varepsilon) > 0$ (this is possible, since $a_1 < a_b$ and $\tau > a_{\dagger} - a_2 + a_1$).

Let us prove that there exists a feedback control u and a corresponding unique nonnegative solution p of (10) satisfying

$$p(\tau, a) = 0$$
 a.e. $a \in (0, a_{\dagger})$.

For almost every $(t, a) \in Q_{\tau}$, let us consider the feedback control u(t, a) := -v(t, a)p(t, a), where

$$v(t,a) := \begin{cases} \frac{1}{a_2 - a} & \text{if } \begin{cases} \theta(\varepsilon) - a_2 \leqslant t - a < \tau - a_1 - a_2 - \varepsilon, \\ a \in [a_1, a_2], \\ \tau - a_1 - a_2 - \varepsilon \leqslant t - a < \tau - a_1 - a_b - \varepsilon, \\ a \in [a_1, a_2], \end{cases}$$

$$v(t,a) := \begin{cases} \frac{1}{\tau - (a_1 + \varepsilon) - t} & \text{if } \\ \frac{1}{a_b - a} & \text{if } \\ \frac{1}{\tau - t} & \text{if } \\ \frac{1}{\tau - t} & \text{if } \\ 0 & \text{otherwise.} \end{cases}$$

$$f(t) = \frac{\theta(\varepsilon) - a_2 \leqslant t - a < \tau - a_1 - a_b - \varepsilon, \\ a \in [a_1, a_2], \\ \tau - a_1 - a_b - \varepsilon \leqslant t - a < \tau - a_b, \\ a \in [a_1, a_b], \\ \alpha \in [a_1, a_b], \\ a \in [a_1, a_b], \end{cases}$$
(21)

The structure of v is given in Fig. 2. The solution to (10) with u = -vp exists, is unique and nonnegative (see Aniţa [5]), and via the comparison principle for agedependent population dynamics [see Theorem 4 (ii)] we get that

$$0 \leqslant p(t,a) \leqslant \tilde{p}(t,a) \quad \text{a.e.} \quad (t,a) \in Q_{\tau}, \tag{22}$$

where \tilde{p} is the solution to

$$\begin{cases} \frac{\partial \tilde{p}}{\partial t}(t,a) + \frac{\partial \tilde{p}}{\partial a}(t,a) + v(t,a)\tilde{p}(t,a) = 0, & (t,a) \in Q_{\tau} \\ \tilde{p}(t,0) = \int_{a_b}^{a_{\dagger}} \beta_0(t,a)\tilde{p}(t,a) \,\mathrm{d}a, & t \in (0,\tau) \\ \tilde{p}(0,a) = p_0(a), & a \in (0,a_{\dagger}). \end{cases}$$
(23)

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Fig. 2 The structure of the control v when $\tau > a_{\dagger} - a_2 + a_1$ and $a_1 < a_b < a_2$

We may discuss three subcases, depending on the order relationship of $\tau - a_1 - a_b - \varepsilon$, $\tau - a_1 - a_2 - \varepsilon$ and 0 [since the expression of the solution \tilde{p} of (23) may be different in each subcase]. The first subcase (see below) is shown in Fig. 2.

Subcase 1 Assume that $\tau - a_1 - a_b - \varepsilon \leq 0$. Integrating along the characteristic lines, we get that the solution \tilde{p} of (23) satisfies

$$\tilde{p}(t,a) = p_0(a-t) \exp\left(-\int_{\max(a-t,a_1)}^{a} \frac{\mathrm{d}s}{a_2-s}\right)$$

if $\theta(\varepsilon) - a_2 \leq t - a < \tau - a_1 - a_2 - \varepsilon$, $a \in [a_1, a_2]$,

$$\tilde{p}(t,a) = p_0(a-t) \exp\left(-\int_{\max(t-a+a_1,0)}^t \frac{\mathrm{d}s}{\tau - (a_1 + \varepsilon) - s}\right)$$

if $\tau - a_1 - a_2 - \varepsilon \leq t - a < \tau - a_1 - a_b - \varepsilon$, $a \in [a_1, a_2]$,

$$\tilde{p}(t,a) = p_0(a-t) \exp\left(-\int_{\max(a-t,a_1)}^{a} \frac{\mathrm{d}s}{a_b-s}\right)$$

if $\tau - a_1 - a_b - \varepsilon \leq t - a \leq 0, \ a \in [a_1, a_b],$

$$\tilde{p}(t,a) = \tilde{p}(t-a,0) \exp\left(-\int_{a_1}^a \frac{\mathrm{d}s}{a_b-s}\right)$$

 $\text{if } 0 < t-a \leqslant \tau - a_b, \ a \in [a_1, a_b], \\$

$$\tilde{p}(t,a) = \tilde{p}(t-a,0) \exp\left(-\int_{t-a+a_1}^t \frac{\mathrm{d}s}{\tau-s}\right)$$

if $\tau - a_b < t - a \leq \tau - (a_1 + \varepsilon)$, $a \in [a_1, a_b]$, so that we have

$$\tilde{p}(t,a) = \frac{a_2 - a}{a_2 - \max(a - t, a_1)} p_0(a - t)$$

if $\theta(\varepsilon) - a_2 \leq t - a < \tau - a_1 - a_2 - \varepsilon$, $a \in [a_1, a_2]$,

$$\tilde{p}(t,a) = \frac{\tau - (a_1 + \varepsilon) - t}{\tau - (a_1 + \varepsilon) - \max(t - a + a_1, 0)} p_0(a - t)$$

if $\tau - a_1 - a_2 - \varepsilon \leq t - a < \tau - a_1 - a_b - \varepsilon$, $a \in [a_1, a_2]$,

$$\tilde{p}(t,a) = \frac{a_b - a}{a_b - \max(a - t, a_1)} p_0(a - t)$$

if $\tau - a_1 - a_b - \varepsilon \leq t - a \leq 0, \ a \in [a_1, a_b],$

$$\tilde{p}(t,a) = \frac{a_b - a}{a_b - a_1} \tilde{p}(t - a, 0)$$

 $\text{if } 0 < t-a \leqslant \tau - a_b, \ a \in [a_1, a_b],$

$$\tilde{p}(t,a) = \frac{\tau - t}{\tau - (t - a + a_1)} \tilde{p}(t - a, 0)$$

if $\tau - a_b < t - a \leq \tau - (a_1 + \varepsilon), \ a \in [a_1, a_b].$

Subcase 2 Assume that $\tau - a_1 - a_2 - \varepsilon \leq 0 < \tau - a_1 - a_b - \varepsilon$. Integrating along the characteristic lines, we get that the solution \tilde{p} of (23) satisfies

$$\tilde{p}(t,a) = p_0(a-t) \exp\left(-\int_{\max(a-t,a_1)}^a \frac{\mathrm{d}s}{a_2-s}\right)$$

 $\text{if }\theta(\varepsilon)-a_2\leqslant t-a<\tau-a_1-a_2-\varepsilon,\ a\in[a_1,a_2],$

$$\tilde{p}(t,a) = p_0(a-t) \exp\left(-\int_{\max(t-a+a_1,0)}^t \frac{\mathrm{d}s}{\tau - (a_1 + \varepsilon) - s}\right)$$

if $\tau - a_1 - a_2 - \varepsilon \leq t - a \leq 0, \ a \in [a_1, a_2],$

$$\tilde{p}(t,a) = \tilde{p}(t-a,0) \exp\left(-\int_{t-a+a_1}^t \frac{\mathrm{d}s}{\tau - (a_1 + \varepsilon) - s}\right)$$

if $0 < t - a \leq \tau - a_1 - a_b - \varepsilon$, $a \in [a_1, a_2]$,

$$\tilde{p}(t, a) = \tilde{p}(t - a, 0) \exp\left(-\int_{a_1}^{a} \frac{\mathrm{d}s}{a_b - s}\right)$$

if $\tau - a_1 - a_b - \varepsilon < t - a \leq \tau - a_b, \ a \in [a_1, a_b],$

$$\tilde{p}(t,a) = \tilde{p}(t-a,0) \exp\left(-\int_{t-a+a_1}^t \frac{\mathrm{d}s}{\tau-s}\right)$$

if $\tau - a_b < t - a \leqslant \tau - (a_1 + \varepsilon)$, $a \in [a_1, a_b]$, so that we have

$$\tilde{p}(t,a) = \frac{a_2 - a}{a_2 - \max(a - t, a_1)} p_0(a - t)$$

 $\text{if }\theta(\varepsilon)-a_2\leqslant t-a<\tau-a_1-a_2-\varepsilon,\ a\in[a_1,a_2],$

$$\tilde{p}(t,a) = \frac{\tau - (a_1 + \varepsilon) - t}{\tau - (a_1 + \varepsilon) - \max(t - a + a_1, 0)} p_0(a - t)$$

if $\tau - a_1 - a_2 - \varepsilon \leq t - a \leq 0, \ a \in [a_1, a_2],$

$$\tilde{p}(t,a) = \frac{\tau - (a_1 + \varepsilon) - t}{\tau - (a_1 + \varepsilon) - (t - a + a_1)} \tilde{p}(t - a, 0)$$

 $\text{if } 0 < t-a \leqslant \tau - a_1 - a_b - \varepsilon, \ a \in [a_1, a_2], \\$

$$\tilde{p}(t,a) = \frac{a_b - a}{a_b - a_1} \tilde{p}(t - a, 0)$$

if $\tau - a_1 - a_b - \varepsilon < t - a \leq \tau - a_b, \ a \in [a_1, a_b],$

$$\tilde{p}(t,a) = \frac{\tau - t}{\tau - (t - a + a_1)} \tilde{p}(t - a, 0)$$

if $\tau - a_b < t - a \leq \tau - (a_1 + \varepsilon), \ a \in [a_1, a_b].$

Subcase 3 Assume that $\tau - a_1 - a_2 - \varepsilon > 0$. Integrating along the characteristic lines, we get that the solution \tilde{p} of (23) satisfies

$$\tilde{p}(t,a) = p_0(a-t) \exp\left(-\int_{\max(a-t,a_1)}^{a} \frac{\mathrm{d}s}{a_2-s}\right)$$

if $\theta(\varepsilon) - a_2 \leq t - a \leq 0, \ a \in [a_1, a_2],$

$$\tilde{p}(t,a) = \tilde{p}(t-a,0) \exp\left(-\int_{a_1}^a \frac{\mathrm{d}s}{a_2-s}\right)$$

 $\text{if } 0 < t-a \leqslant \tau - a_1 - a_2 - \varepsilon, \ a \in [a_1, a_2],$

$$\tilde{p}(t,a) = \tilde{p}(t-a,0) \exp\left(-\int_{t-a+a_1}^t \frac{\mathrm{d}s}{\tau - (a_1 + \varepsilon) - s}\right)$$

if $\tau - a_1 - a_2 - \varepsilon < t - a \leqslant \tau - a_1 - a_b - \varepsilon$, $a \in [a_1, a_2]$,

$$\tilde{p}(t,a) = \tilde{p}(t-a,0) \exp\left(-\int_{a_1}^{a} \frac{\mathrm{d}s}{a_b-s}\right)$$

if $\tau - a_1 - a_b - \varepsilon < t - a \leqslant \tau - a_b, \ a \in [a_1, a_b],$

$$\tilde{p}(t,a) = \tilde{p}(t-a,0) \exp\left(-\int_{t-a+a_1}^t \frac{\mathrm{d}s}{\tau-s}\right)$$

if $\tau - a_b < t - a \leq \tau - (a_1 + \varepsilon)$, $a \in [a_1, a_b]$, so that we have

$$\tilde{p}(t,a) = \frac{a_2 - a}{a_2 - \max(a - t, a_1)} p_0(a - t)$$

 $\text{if }\theta(\varepsilon)-a_2\leqslant t-a\leqslant 0,\;a\in[a_1,a_2],\\$

$$\tilde{p}(t,a) = \frac{a_2 - a}{a_2 - a_1} \tilde{p}(t - a, 0)$$

 $\text{if } 0 < t-a \leqslant \tau - a_1 - a_2 - \varepsilon, \ a \in [a_1, a_2], \\$

$$\tilde{p}(t,a) = \frac{\tau - (a_1 + \varepsilon) - t}{\tau - (a_1 + \varepsilon) - (t - a + a_1)} \tilde{p}(t - a, 0)$$

if $\tau - a_1 - a_2 - \varepsilon < t - a \leqslant \tau - a_1 - a_b - \varepsilon$, $a \in [a_1, a_2]$,

$$\tilde{p}(t,a) = \frac{a_b - a}{a_b - a_1} \tilde{p}(t - a, 0)$$

if $\tau - a_1 - a_b - \varepsilon < t - a \leq \tau - a_b, \ a \in [a_1, a_b],$

$$\tilde{p}(t,a) = \frac{\tau - t}{\tau - (t - a + a_1)} \tilde{p}(t - a, 0)$$

if $\tau - a_b < t - a \leq \tau - (a_1 + \varepsilon), \ a \in [a_1, a_b].$

In the three subcases, we may conclude that we have

$$\begin{cases}
\tilde{p}(t, a_2) = 0 & \text{if } t \in (\theta(\varepsilon), \tau - (a_1 + \varepsilon)), \\
\tilde{p}(\tau - (a_1 + \varepsilon), a) = 0 & \text{if } a \in (a_b, a_2), \\
\tilde{p}(t, a_b) = 0 & \text{if } t \in (\tau - (a_1 + \varepsilon), \tau), \\
\tilde{p}(\tau, a) = 0 & \text{if } a \in (a_1 + \varepsilon, a_b).
\end{cases}$$
(24)

Integrating on the characteristic lines the equation satisfied by \tilde{p} , we get that

$$\tilde{p}(t,a) = 0 \quad \text{if } (t,a) \in [\tau - (a_1 + \varepsilon), \tau] \times [a_b, a_{\dagger}]. \tag{25}$$

Since \tilde{p} satisfies the renewal law, we may infer that

$$\tilde{p}(t,0) = 0$$
 if $t \in [\tau - (a_1 + \varepsilon), \tau]$,

and integrating along the characteristic lines we get that

$$\tilde{p}(\tau, a) = 0 \quad \text{if } a \in [0, a_1 + \varepsilon]. \tag{26}$$

By (24), (25) and (26) we get that

$$\tilde{p}(\tau, a) = 0$$
 a.e. $a \in (0, a_{\dagger})$.

It remains to prove that, for each subcase, the control u = -vp belongs to $L^2(Q_\tau)$. In order to avoid repetitive arguments, we prove that $u \in L^2(Q_\tau)$ only in the first subcase. In this subcase, for almost every $(t, a) \in Q_\tau$, from (21) and the corresponding expression of \tilde{p} we have that $v\tilde{p}$ is equal to

$$\begin{cases} \frac{p_{0}(a-t)}{a_{2}-\max(a-t,a_{1})} & \text{if} \\ \frac{p_{0}(a-t)}{\tau-(a_{1}+\varepsilon)-\max(t-a+a_{1},0)} & \text{if} \\ \frac{p_{0}(a-t)}{\tau-(a_{1}+\varepsilon)-\max(t-a+a_{1},0)} & \text{if} \\ \frac{p_{0}(a-t)}{a_{b}-\max(a-t,a_{1})} & \text{if} \\ \frac{\tilde{p}(t-a,0)}{a_{b}-a_{1}} & \text{if} \\ \frac{\tilde{p}(t-a,0)}{\tau-(t-a+a_{1})} & \text{if} \\ \frac{\tilde{p}(t-a,0)}{\tau-(t-a+a_{1})} & \text{if} \\ \frac{\tilde{p}(t-a,0)}{\tau-(t-a+a_{1})} & \text{if} \\ \frac{1}{2} & \frac{1}{2} - \frac{1}{2} - \frac{1}{2} \leq t-a \leq \tau - a \leq \tau - a_{1} - a_{b} - \varepsilon, \\ a \in [a_{1}, a_{b}], \\ 0 < t - a \leq \tau - a_{b}, \\ a \in [a_{1}, a_{b}], \\ \tau - a_{b} < t - a \leq \tau - (a_{1} + \varepsilon), \\ a \in [a_{1}, a_{b}], \\ 0 & \text{otherwise}, \end{cases}$$

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and it follows that we have

$$|v\tilde{p}| \leqslant \begin{cases} \frac{|p_0(a-t)|}{\min(\theta(\varepsilon),a_2-a_1)} & \text{if} \\ \frac{|p_0(a-t)|}{\min(\tau-(a_1+\varepsilon),a_b-a_1)} & \text{if} \\ \frac{|p_0(a-t)|}{\min(\tau-(a_1+\varepsilon),a_b-a_1)} & \text{if} \\ \frac{|p_0(a-t)|}{\min(\tau-(a_1+\varepsilon),a_b-a_1)} & \text{if} \\ \frac{|\tilde{p}(t-a,0)|}{a_b-a_1} & \text{if} \\ \frac{|\tilde{p}(t-a,0)|}{\varepsilon} & \text{if} \\ \frac{|\tilde{p}(t-a,0)|}{\varepsilon} & \text{if} \\ \frac{|\tilde{p}(t-a,0)|}{\varepsilon} & \text{if} \\ 0 < t-a \leqslant \tau - a_b, \\ a \in [a_1, a_b], \\ 0 < t-a \leqslant \tau - a_b, \\ a \in [a_1, a_b], \\ 0 < t-a \leqslant \tau - (a_1+\varepsilon), \\ a \in [a_1, a_b], \\ 0 < t-a \leqslant \tau - (a_1+\varepsilon), \\ a \in [a_1, a_b], \end{cases}$$

Since $\tilde{p} \in L^2(Q_\tau)$, we may infer from the renewal law that we have $\tilde{p}(\cdot, 0) \in L^2(0, \tau)$, so that we have $v\tilde{p} \in L^2(Q_\tau)$. Given (22), it follows that we have $vp \in L^2(Q_\tau)$, which concludes the proof of Proposition 1 in the case where $a_2 > a_b$.

3.3 Proof of Theorem 1: from the linear to the nonlinear case

This subsection is devoted to prove Theorem 1, using the result of Proposition 1 and the comparison principle for age-structured population dynamics (see Theorem 4).

Proof of Theorem 1 Assume that there exists $a_b > 0$ such that (3) holds. Suppose that $a_1 < a_b$, and let $\tau > a_{\dagger} - a_2 + a_1$. Suppose, first, that $a_2 \leq a_b$. Denote by v the function defined by (13). The solution p to (1) with u := -vp exists, is unique, and via the comparison result for age-dependent population dynamics, we get that

$$0 \leqslant p(t,a) \leqslant \overline{p}(t,a) \quad \text{a.e.} \ (t,a) \in Q_{\tau}, \tag{27}$$

where \overline{p} is solution to the linear system:

$$\begin{cases} \frac{\partial \overline{p}}{\partial t}(t,a) + \frac{\partial \overline{p}}{\partial a}(t,a) \\ +(\mu(t,a,0) + v(t,a)\chi_{[a_1,a_2]}(a))\overline{p}(t,a) = 0, \quad (t,a) \in Q_{\tau} \\ \overline{p}(t,0) = \int_{0}^{a_{\dagger}} \beta_{+}(t,a)\overline{p}(t,a) \, \mathrm{d}a, \qquad t \in (0,\tau) \\ \overline{p}(0,a) = p_{0}(a), \qquad a \in (0,a_{\dagger}). \end{cases}$$
(28)

From the proof of Proposition 1, we get that the solution \overline{p} of (28) satisfies $\overline{p}(\tau, \cdot) = 0$, with $v\overline{p} \in L^2(Q_\tau)$, so that using (27) the solution p to (1) with u = -pv satisfies $p(\tau, \cdot) = 0$ with $u \in L^2(Q_\tau)$.

The case where $a_b < a_2$ is similar, denoting by v the function defined by (21). This concludes the proof of Theorem 1.

4 Negative null controllability results when the young individuals do not reproduce

This section is devoted to prove that the results obtained in Theorem 1 are sharp, in the sense that if we do not control before the individuals start to reproduce or if the control time horizon τ is too small, then system (1) may not be null controllable. Following the methodology of the previous Sect. 1, we first prove Theorem 2 in the linear setting (see Proposition 2) and we shall deduce the results of Theorem 2 from those of Proposition 2 and the comparison principle for age-structured population dynamics (see Theorem 4).

More precisely, let us first consider the linear controlled system (10) subject to assumptions (L1)-(L4) stated in the beginning of Sect. 3. We first prove the following Proposition 2 which is the "linear version" of Theorem 2:

Proposition 2 With the above notations and assumptions,

(i) Assume that there exists $a_b \in (0, a_{\dagger})$ and $a_m \in (0, a_{\dagger}]$ with $a_b < a_m$ such that, for almost every $t \in (0, \tau)$ we have

$$\beta_0(t,a) > 0 \quad a.e. \ a \in (a_b, a_m).$$
 (29)

If $a_1 > a_b$, then for every $\tau > 0$, there exists $p_0 \in L^2(0, a_{\dagger})$ with $p_0(a) \ge 0$ a.e. $a \in (0, a_{\dagger})$ such that there is no control u and a corresponding nonnegative solution p to (10) satisfying

$$p(\tau, a) = 0$$
 a.e. $a \in (0, a_{\dagger})$.

(ii) If $\tau < a_{\dagger} - a_2 + a_1$, then there exists a function β_0 satisfying (L1)–(L2) and an initial population density $p_0 \in L^2(0, a_{\dagger})$ with $p_0(a) \ge 0$ a.e. $a \in (0, a_{\dagger})$ such that there is no control u and a corresponding nonnegative solution p to (10) satisfying

$$p(\tau, a) = 0$$
 a.e. $a \in (0, a_{\dagger})$.

Proof of Proposition 2(i) Suppose that there exists $a_b \in (0, a_{\dagger})$ and $a_m \in (0, a_{\dagger}]$ with $a_b < a_m$ such that (29) holds. Assume, first, that $a_1 = a_b$ and let $\tau > 0$. Since $a_1 > 0$, note that there exists $m \in \mathbb{N}$ and $r \in [0, a_1)$ such that $\tau = ma_1 + r$. We may suppose, without loss of generality, that $m \ge 2$. Let $p_0 \in L^2(0, a_{\dagger})$ such that

$$p_0(a) > 0$$
 a.e. $a \in (0, a_1 - r)$. (30)

Suppose that there exists a control function $u \in L^2(Q_\tau)$ and a corresponding nonnegative solution p of (10) such that

$$p(\tau, a) = 0$$
 a.e. $a \in (0, a_{\dagger})$

Since $p(\tau, a) = 0$ for almost every $a \in (0, a_1)$, integrating along the characteristic lines we have that

$$p(t, 0) = 0$$
 a.e. $t \in (\tau - a_1, \tau)$. (31)

Since *p* satisfies the renewal law, for almost every $t \in (\tau - a_1, \tau)$ we have that

$$p(t,0) = \int_0^{a_{\uparrow}} \beta_0(t,a) p(t,a) \, \mathrm{d}a \ge \int_{a_b}^{a_m} \beta_0(t,a) p(t,a) \, \mathrm{d}a.$$
(32)

Since $p(t, a) \ge 0$ for almost every $(t, a) \in Q_{\tau}$, using (31) and (32) we have, for almost every $t \in (\tau - a_1, \tau)$,

$$\int_{a_b}^{a_m} \beta_0(t, a) p(t, a) \, \mathrm{d}a = 0.$$
(33)

Using the assumptions on the function β_0 and the fact that $p(t, a) \ge 0$ for almost every $(t, a) \in Q_{\tau}$, together with the above equality (33) we have that

$$p(t, a) = 0$$
 a.e. $(t, a) \in (\tau - a_1, \tau) \times (a_b, a_m)$.

Integrating along the characteristic lines, we have that

$$p(t, 0) = 0$$
 a.e. $t \in (\tau - 2a_1, \tau - a_1)$.

Using and induction argument, we may infer that

$$p(t, 0) = 0$$
 a.e. $t \in (\tau - (m - 1)a_1, \tau - ma_1)$.

Recall that $\tau - ma_1 = r$. Using again the renewal law, it follows from the above equality that

$$p(t, a) = 0$$
 a.e. $(t, a) \in (\tau - (m - 1)a_1, r) \times (a_b, a_m),$

and integrating along the characteristic lines we have that

$$p(0, a) = 0$$
 a.e. $a \in (0, a_1 - r)$,

which is a contradiction with (30) (see Fig. 3).

Since it is not possible to bring the population to zero in any time τ in the case where $a_1 = a_b$, we may infer that it is not possible to bring the population to zero in the case where $a_1 > a_b$, so that Proposition 2(i) is proved.



Fig. 3 In this figure, we have $\tau = 3a_1 + r$ with $r \in [0, a_1)$. If we suppose that the population vanishes in time τ , we may deduce that the population vanishes in the whole blue dashed domain, so that the initial population has to be null in the age interval $(0, a_1 - r)$

Proof of Proposition 2(ii) Let $\tau < a_{\dagger} - a_2 + a_1$. Suppose that there exists a control function $u \in L^2(Q_{\tau})$ and a corresponding nonnegative solution p of (10) such that

$$p(\tau, a) = 0$$
 a.e. $a \in (0, a_{\dagger})$.

We may divide this proof in two cases.

Case 1 Suppose that $\tau < a_{\dagger} - a_2$. Then, for almost every $(t, a) \in Q_{\tau}$ with $a_2 \leq a - t \leq a_{\dagger} - \tau$, we have that

$$\int_0^t \exp\left\{-\int_s^t \mu_0(\sigma, a - t + \sigma) \,\mathrm{d}\sigma\right\} \chi_{[a_1, a_2]}(a - t + s)u(s, a - t + s) \,\mathrm{d}s = 0,$$

so that from (11), for almost every $(t, a) \in Q_{\tau}$ with $a_2 \leq a - t \leq a_{\dagger} - \tau$ we have that

$$p(t, a) = \exp\left\{-\int_0^t \mu_0(s, a - t + s) \,\mathrm{d}s\right\} p_0(a - t),$$

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and it follows from the above equality that we have that

$$p(\tau, a) = \exp\left\{-\int_0^\tau \mu_0(s, a - \tau + s) \,\mathrm{d}s\right\} p_0(a - \tau) \quad \text{a.e. } a \in (\tau + a_2, a_\dagger),$$

so that from the above equation the condition $p(\tau, \cdot) = 0$ cannot be fulfilled in the case where $p_0(a) \neq 0$ for almost every $a \in (a_2, a_{\dagger} - \tau)$.

Case 2 Suppose that $\tau \ge a_{\dagger} - a_2$. Since $p(\tau, a) = 0$ for almost every $a \in (0, a_1)$, integrating along the characteristic lines we have that

$$p(t, 0) = 0$$
 a.e. $t \in (\tau - a_1, \tau)$. (34)

Note that, since $\tau < a_{\dagger} - a_2 + a_1$, we have $\tau - a_1 < a_{\dagger} - a_2$ so that from (34) and using the fact that $a_{\dagger} - a_2 \leq \tau$, we have, in particular,

$$p(t, 0) = 0$$
 a.e. $t \in (\tau - a_1, a_{\dagger} - a_2)$.

Since p satisfies the renewal law, for almost every $t \in (\tau - a_1, a_{\dagger} - a_2)$ we have that

$$p(t,0) = \int_0^{a_{\dagger}} \beta_0(t,a) p(t,a) \, \mathrm{d}a = 0.$$
(35)

Since $\beta_0(t, a) \ge 0$ and $p(t, a) \ge 0$ for almost every $(t, a) \in (\tau - a_1, a_{\dagger} - a_2) \times (0, a_{\dagger})$, we deduce from the above equality (35) that we have

$$\beta_0(t, a) p(t, a) = 0 \quad \text{a.e.} \ (t, a) \in (\tau - a_1, a_{\dagger} - a_2) \times (0, a_{\dagger}). \tag{36}$$

Define the triangle

$$\mathcal{T} := \{(t, a) \in Q_{\tau}, a_2 \leqslant a - t \leqslant a_{\dagger} - (\tau - a_1), a \geqslant \tau - a_1 + a_2, t \geqslant \tau - a_1(\beta 7)\}$$

See Fig. 4 for a picture of the triangle position. From (36) and (37), for almost every $(t, a) \in \mathcal{T}$ we have

$$\beta_0(t, a) p(t, a) = 0.$$
(38)

Moreover, for almost every $(t, a) \in \mathcal{T}$ we have that

$$\int_0^t \exp\left\{-\int_s^t \mu_0(\sigma, a - t + \sigma) \,\mathrm{d}\sigma\right\} \chi_{[a_1, a_2]}(a - t + s)u(s, a - t + s) \,\mathrm{d}s = 0,$$

so that from (11), for almost every $(t, a) \in \mathcal{T}$ we have that

$$p(t, a) = \exp\left\{-\int_0^t \mu_0(s, a - t + s) \,\mathrm{d}s\right\} p_0(a - t),$$

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Fig. 4 In this figure, we have $\tau < a_{\dagger} - a_2 + a_1$ and $\tau \ge a_{\dagger} - a_2$. If we suppose that the population vanishes in time τ , because of the renewal law, we may deduce that the population or the function β has to be null within the blue dashed triangle \mathcal{T} (region which is not influenced by the control). This is not the case in general : We may choose an initial condition which is strictly positive in the age interval $(a_2, a_{\dagger} - (\tau - a_1))$ and a strictly positive function β in the blue dashed triangle

and using the above equality, the condition (38) may be rewritten as

$$\beta_0(t,a) \exp\left\{-\int_0^t \mu_0(s,a-t+s) \,\mathrm{d}s\right\} p_0(a-t) = 0 \quad \text{a.e.} \ (t,a) \in \mathcal{T}.$$

This condition may not be fulfilled in general: take, for instance, $\beta_0(t, a) = 1$ for almost every $(t, a) \in \mathcal{T}$ and $p_0(a) > 0$ for almost every $a \in (a_2, a_{\dagger} - (\tau - a_1))$. This concludes the proof of Proposition 2(ii).

We now prove Theorem 2 in the more general nonlinear case, using again the comparison principle and the results of Proposition 2 relatively to the linear setting.

Proof of Theorem 2 (i) Let $\tau > 0$ and $a_1 > 0$. Assume by contradiction that there exists a control u and a corresponding unique nonnegative solution p of (1) such that $p(\tau, a) = 0$ a.e. $a \in (0, a_{\dagger})$. We have that $P \in L^{\infty}(0, \tau)$, where $P(t) = \int_{0}^{a_{\dagger}} p(t, a) da$. Denote by $M = ||P||_{L^{\infty}(0, \tau)}$ and by

$$\beta_{-}(t, a) := \beta(t, a, M), \quad \mu_{+}(t, a) := \mu(t, a, 0), \quad \text{a.e.} \ (t, a) \in Q_{\tau}.$$

By the comparison principle for age-structured population dynamics we get that

$$0 \le \tilde{p}(t,a) \le p(t,a), \quad (t,a) \in Q^1_\tau$$

(along almost every characteristic line), where $Q_{\tau}^1 = (0, \tau) \times (0, \overline{a}_1), \overline{a}_1 = \min\{a_1, a_m\}$, and \tilde{p} is the solution to the linear system

$$\begin{cases} \frac{\partial \tilde{p}}{\partial t}(t,a) + \frac{\partial \tilde{p}}{\partial a}(t,a) + \mu_{+}(t,a)\tilde{p}(t,a) = 0, & (t,a) \in Q_{\tau}^{1} \\ \tilde{p}(t,0) = \int_{0}^{\overline{a}_{1}} \beta_{-}(t,a)\tilde{p}(t,a) \,\mathrm{d}a, & t \in (0,\tau) \\ \tilde{p}(0,a) = p_{0}(a), & a \in (0,\overline{a}_{1}). \end{cases}$$

More precisely, we have used the comparison principle [see Theorem 4(ii)] for the functions β_1 , β_2 , μ_1 and μ_2 defined on Q_{τ} by

$$\beta_1(t,a) := \beta_-(t,a), \quad \beta_2(t,a) := \begin{cases} \beta_-(t,a) & \text{if } (t,a) \in Q_\tau^1, \\ 0 & \text{otherwise}, \end{cases}$$
$$\mu_1(t,a) := \mu_+(t,a), \quad \mu_2(t,a) := \mu(t,a,P(t)),$$

noting that we have

$$\beta_1(t,a) \ge \beta_2(t,a)$$
 and $\mu_1(t,a) \le \mu_2(t,a)$ a.e. in Q_{τ} .

Since $\beta_{-}(t, a) > 0$ a.e. $(t, a) \in (0, \tau) \times (a_b, \overline{a}_1)$, we may conclude that for p_0 satisfying

$$p_0(a) > 0$$
 a.e. $a \in (0, \overline{a}_1)$,

we get that $\tilde{p}(t, a) > 0$ along almost any characteristic line. Since we have $\tilde{p}(\tau, a) > 0$ a.e. $a \in (0, \overline{a}_1)$, we get a contradiction with the fact that $p(\tau, a) = 0$ a.e. $a \in (0, \overline{a}_1)$.

Proof of Theorem 2(ii) Let us take $\beta := \beta_0$, where β_0 is chosen as at the end of the proof of Proposition 2(ii), then arguing again by contradiction and using the comparison principle for age-structured population dynamics we get the conclusion as in Theorem 2(i).

5 Controllability results with general assumptions on the fertility rate

This section is devoted to prove Theorem 3, when no extra assumption on the function β is assumed (in particular, we may have $\beta > 0$ in some interval of ages near zero). To this aim, we follow the methodology of the two previous sections and we first prove Theorem 3 in the linear setting. The results of Theorem 3 will be deduced from those of Proposition 3 and by the comparison principle.

Let us first consider the linear controlled system (10) subject to assumptions (L1)–(L4) stated in Sect. 3. We will prove the following Proposition 3 which is the "linear version" of Theorem 3:

Proposition 3 With the above notations and assumptions,

(i) Assume that $a_1 = 0$ and $a_2 > 0$. If $\tau > a_{\dagger} - a_2$, then for every $p_0 \in L^2(0, a_{\dagger})$ with $p_0(a) \ge 0$ a.e. $a \in (0, a_{\dagger})$, there exists $u \in L^2(Q_{\tau})$ and a corresponding unique nonnegative solution p of (10) such that

$$p(\tau, a) = 0$$
 a.e. $a \in (0, a_{\dagger})$.

(ii) Assume that there exists $a_m \in (0, a_{\dagger}]$ such that, for almost every $t \in (0, \tau)$ we have

$$\beta_0(t, a) > 0$$
 a.e. $a \in (0, a_m)$.

If $a_1 > 0$, then for every $\tau > 0$, there exists $p_0 \in L^2(0, a_{\dagger})$ with $p_0(a) \ge 0$ a.e. $a \in (0, a_{\dagger})$ such that there is no control u and a corresponding nonnegative solution p to (10) satisfying

$$p(\tau, a) = 0$$
 a.e. $a \in (0, a_{\dagger})$.

(iii) If $\tau < a_{\dagger} - a_2$, there exists an initial population density $p_0 \in L^2(0, a_{\dagger})$ with $p_0(a) \ge 0$ a.e. $a \in (0, a_{\dagger})$ such that there is no control u and a corresponding nonnegative solution p to (10) satisfying

$$p(\tau, a) = 0$$
 a.e. $a \in (0, a_{\dagger})$.

Proof of Proposition 3(i) Suppose that $\tau > a_{\dagger} - a_2$. Without loss of generality, we may assume that $\tau < a_{\dagger}$. Let us prove that there exists a feedback control u and a corresponding unique nonnegative solution p of (10) such that

$$p(\tau, a) = 0$$
 a.e. $a \in (0, a_{\dagger})$.

Denote by $\varepsilon := \frac{1}{2}(\tau - (a_{\dagger} - a_2))$. For almost every $(t, a) \in Q_{\tau}$, let us consider the feedback control u(t, a) := -v(t, a)p(t, a), where

$$v(t,a) := \begin{cases} \frac{1}{a_2 - a} & \text{if} \begin{cases} \varepsilon - a_2 \leqslant t - a < a_{\dagger} - 2a_2 + \varepsilon, \\ a \in [0, a_2], \\ a_{\dagger} - a_2 + \varepsilon - t \end{cases} & \text{if} \begin{cases} a_{\dagger} - 2a_2 + \varepsilon \leqslant t - a < a_{\dagger} - 2a_2 + 2\varepsilon, \\ a \in [0, a_2], \\ a_{\dagger} - 2a_2 + 2\varepsilon \leqslant t - a < a_{\dagger} - a_2 + \varepsilon, \\ a \in [0, a_2], \\ a_{\dagger} - 2a_2 + 2\varepsilon \leqslant t - a < a_{\dagger} - a_2 + \varepsilon, \\ t \in [a_{\dagger} - 2a_2 + 2\varepsilon, a_{\dagger} - a_2 + \varepsilon], \\ a \in [0, a_2 - \varepsilon], \\ a \in [0, a_2 - \varepsilon], \\ a_{\dagger} - a_2 + \varepsilon \leqslant t \leqslant \tau, \\ a \in [0, a_2], \\ 0 & \text{otherwise.} \end{cases}$$
(39)

The structure of v is given in Fig. 5. The solution to (10) with u = -vp exists, is unique and nonnegative (see Anita [5]), and via the comparison principle for age-



Fig. 5 The structure of the control v when $a_1 = 0$ and $\tau > a_{\dagger} - a_2$

dependent population dynamics [see Theorem 4(ii)] we get that

$$0 \leqslant p(t,a) \leqslant \tilde{p}(t,a) \quad \text{a.e.} \quad (t,a) \in Q_{\tau}, \tag{40}$$

where \tilde{p} is the solution to

$$\begin{cases} \frac{\partial \tilde{p}}{\partial t}(t,a) + \frac{\partial \tilde{p}}{\partial a}(t,a) + v(t,a)\tilde{p}(t,a) = 0, & (t,a) \in Q_{\tau} \\ \tilde{p}(t,0) = M \int_{0}^{a_{\dagger}} \tilde{p}(t,a) \, \mathrm{d}a, & t \in (0,\tau) \\ \tilde{p}(0,a) = p_{0}(a), & a \in (0,a_{\dagger}), \end{cases}$$
(41)

where $M := \|\beta_0\|_{L^{\infty}(Q_{\tau})}$.

There may be again three subcases to discuss, depending on the order relationship of $a_{\dagger} - 2a_2 + \varepsilon$, $a_{\dagger} - 2a_2 + 2\varepsilon$ and 0 (since the expression of the solution \tilde{p} of (41) may be different in each subcase). Here, we do the proof only in the case when $a_{\dagger} - 2a_2 + \varepsilon > 0$ (namely, the case when a_2 is sufficiently small) which is pictured in Fig. 5. In this case, integrating along the characteristic lines, we get that the solution \tilde{p} of (41) satisfies

$$\tilde{p}(t,a) = p_0(a-t) \exp\left(-\int_{a-t}^a \frac{\mathrm{d}s}{a_2-s}\right)$$

if $\varepsilon - a_2 \leq t - a \leq 0, \ a \in [0, a_2],$

$$\tilde{p}(t,a) = \tilde{p}(t-a,0) \exp\left(-\int_0^a \frac{\mathrm{d}s}{a_2-s}\right)$$

if $0 < t - a \leq a_{\dagger} - 2a_2 + \varepsilon$, $a \in [0, a_2]$,

$$\tilde{p}(t,a) = \tilde{p}(t-a,0) \exp\left(-\int_{t-a}^{t} \frac{\mathrm{d}s}{a_{\dagger}-a_{2}+\varepsilon-s}\right)$$

 $\text{if } a_{\dagger} - 2a_2 + \varepsilon < t - a \leqslant a_{\dagger} - 2a_2 + 2\varepsilon, \ a \in [0, a_2],$

$$\tilde{p}(t,a) = \tilde{p}(t-a,0)$$

if $a_{\dagger} - 2a_2 + 2\varepsilon < t - a \leq a_{\dagger} - a_2 + \varepsilon$, $t \in [a_{\dagger} - 2a_2 + 2\varepsilon, a_{\dagger} - a_2 + \varepsilon]$, $a \in [0, a_2 - \varepsilon]$, so that we have

$$\tilde{p} = \begin{cases} \frac{a_2 - a}{a_2 - (a - t)} p_0(a - t) & \text{if} \\ \frac{a_2 - a}{a_2} \tilde{p}(t - a, 0) & \text{if} \\ \frac{a_1 - a_2 + \varepsilon - t}{a_1 - a_2 + \varepsilon - (t - a)} \tilde{p}(t - a, 0) & \text{if} \\ \tilde{p}(t - a, 0) & \text{if} \end{cases} \begin{cases} \varepsilon - a_2 \leqslant t - a \leqslant 0, \\ a \in [0, a_2], \\ 0 < t - a \leqslant a_1 - 2a_2 + \varepsilon, \\ a \in [0, a_2], \\ a_1 - 2a_2 + \varepsilon < t - a \leqslant a_1 - 2a_2 + 2\varepsilon (42) \\ a \in [0, a_2], \\ a_1 - 2a_2 + 2\varepsilon < t - a \leqslant a_1 - a_2 + \varepsilon, \\ t \in [a_1 - 2a_2 + 2\varepsilon, a_1 - a_2 + \varepsilon], \\ a \in [0, a_2 - \varepsilon]. \end{cases}$$

We may conclude from the above formulas that we have

$$\begin{cases} \tilde{p}(t, a_2) = 0 & \text{if } t \in (\varepsilon, a_{\dagger} - a_2 + \varepsilon), \\ \tilde{p}(a_{\dagger} - a_2 + \varepsilon, a) = 0 & \text{if } a \in (a_2 - \varepsilon, a_2). \end{cases}$$
(43)

Integrating on the characteristic lines the equation satisfied by \tilde{p} , we get from (43) that

$$\tilde{p}(t,a) = 0 \quad \text{if } (t,a) \in [a_{\dagger} - a_2 + \varepsilon, \tau] \times [a_2, a_{\dagger}]. \tag{44}$$

By (44) we get that

$$-\frac{1}{\tau - t}\chi_{[0,a_2]}(a)\tilde{p}(t,a) = -\frac{1}{\tau - t}\tilde{p}(t,a)$$

a.e. $(t, a) \in [a_{\dagger} - a_2 + \varepsilon, \tau] \times (0, a_{\dagger})$, and so we may infer that \tilde{p} is the unique nonnegative solution to

$$\begin{cases} \frac{\partial \tilde{p}}{\partial t}(t,a) + \frac{\partial \tilde{p}}{\partial a}(t,a) + \frac{1}{\tau-t}\tilde{p}(t,a) = 0, & (t,a) \in (a_{\dagger} - a_{2} + \varepsilon, \tau) \times (0, a_{\dagger}) \\ \tilde{p}(t,0) = M \int_{0}^{a_{\dagger}} \tilde{p}(t,a) \, \mathrm{d}a, & t \in (a_{\dagger} - a_{2} + \varepsilon, \tau) \\ \tilde{p}(a_{\dagger} - a_{2} + \varepsilon, a) = \tilde{p}(a_{\dagger} - a_{2} + \varepsilon, a), & a \in (0, a_{\dagger}). \end{cases}$$

We may conclude that

$$\tilde{p}(t,a) = h(t)g(t,a)$$

a.e. $(t, a) \in [a_{\dagger} - a_2 + \varepsilon, \tau] \times (0, a_{\dagger})$ (\tilde{p} is separable), where g is the solution to

$$\begin{cases} \frac{\partial g}{\partial t}(t,a) + \frac{\partial g}{\partial a}(t,a) = 0, & (t,a) \in (a_{\dagger} - a_{2} + \varepsilon, \tau) \times (0, a_{\dagger}) \\ g(t,0) = M \int_{0}^{a_{\dagger}} g(t,a) \, \mathrm{d}a, & t \in (a_{\dagger} - a_{2} + \varepsilon, \tau) \\ g(a_{\dagger} - a_{2} + \varepsilon, a) = \tilde{p}(a_{\dagger} - a_{2} + \varepsilon, a), & a \in (0, a_{\dagger}), \end{cases}$$

and h is the solution to

$$\begin{cases} h'(t) + \frac{1}{\tau - t}h(t) = 0, \ t \in (a_{\dagger} - a_2 + \varepsilon, \tau) \\ h(a_{\dagger} - a_2 + \varepsilon) = 1. \end{cases}$$

Hence for every $t \in [a_{\dagger} - a_2 + \varepsilon, \tau]$ we have

$$h(t) = \frac{1}{\tau - (a_{\dagger} - a_2 + \varepsilon)} (\tau - t).$$
(45)

By (45) we get that $\tilde{p}(\tau, a) = 0$ a.e. $a \in (0, a_{\dagger})$ and consequently

$$p(\tau, a) = 0$$

a.e. $a \in (0, a_{\dagger})$.

It remains to prove that the control u = -vp belongs to $L^2(Q_\tau)$. For almost every $(t, a) \in (0, a_{\dagger} - a_2 + \varepsilon) \times (0, a_2)$, from (39) and (42) we have that

$$|v\tilde{p}| = \begin{cases} \frac{p_0(a-t)}{a_2 - (a-t)} & \text{if} \\ \frac{\tilde{p}(t-a,0)}{a_2} & \text{if} \\ \frac{\tilde{p}(t-a,0)}{a_2} & \text{if} \\ \frac{\tilde{p}(t-a,0)}{a_{\dagger} - a_2 + \varepsilon - (t-a)} & \text{if} \\ \frac{a \in [0,a_2],}{a_{\dagger} - 2a_2 + \varepsilon < t - a \leqslant a_{\dagger} - 2a_2 + 2\varepsilon (46)} \\ a \in [0,a_2], \\ a_{\dagger} - 2a_2 + \varepsilon < t - a \leqslant a_{\dagger} - 2a_2 + 2\varepsilon (46) \\ a \in [0,a_2], \\ a_{\dagger} - 2a_2 + 2\varepsilon < t - a \leqslant a_{\dagger} - a_2 + \varepsilon, \\ t \in [a_{\dagger} - 2a_2 + 2\varepsilon, a_{\dagger} - a_2 + \varepsilon], \\ a \in [0,a_2 - \varepsilon], \end{cases}$$

and it follows from (46) that we have

$$|v\tilde{p}| \leqslant \begin{cases} \frac{|p_{0}(a-t)|}{\varepsilon} & \text{if } \begin{cases} \varepsilon - a_{2} \leqslant t - a \leqslant 0, \\ a \in [0, a_{2}], \\ 0 < t - a \leqslant a_{\dagger} - 2a_{2} + \varepsilon, \\ a \in [0, a_{2}], \\ 0 < t - a \leqslant a_{\dagger} - 2a_{2} + \varepsilon, \\ a \in [0, a_{2}], \\ a_{\dagger} - 2a_{2} + \varepsilon < t - a \leqslant a_{\dagger} - 2a_{2} + 2\varepsilon, \\ a \in [0, a_{2}], \\ a_{\dagger} - 2a_{2} + 2\varepsilon < t - a \leqslant a_{\dagger} - a_{2} + \varepsilon, \\ t \in [a_{\dagger} - 2a_{2} + 2\varepsilon, a_{\dagger} - a_{2} + \varepsilon], \\ a \in [0, a_{2} - \varepsilon]. \end{cases}$$
(47)

Since $\tilde{p} \in L^2((0, a_{\dagger} - a_2 + \varepsilon) \times (0, a_{\dagger}))$, we may infer from the renewal law that we have $\tilde{p}(\cdot, 0) \in L^2((0, a_{\dagger} - a_2 + \varepsilon))$, so that from (47) we have $v\tilde{p} \in L^2((0, a_{\dagger} - a_2 + \varepsilon) \times (0, a_{\dagger}))$. Given (40), it follows that we have $vp \in L^2((0, a_{\dagger} - a_2 + \varepsilon) \times (0, a_{\dagger}))$.

On the other hand, for almost every $(t, a) \in (a_{\dagger} - a_2 + \varepsilon) \times (0, a_2)$ we have

$$|v(t,a)p(t,a)| \le |v(t,a)\tilde{p}(t,a)| = \frac{1}{\tau - t}h(t)g(t,a) = \frac{1}{\tau - (a_{\dagger} - a_2 + \varepsilon)}g(t,a),$$

and so that $vp \in L^2((a_{\dagger} - a_2 + \varepsilon, \tau) \times (0, a_2))$, which ends the proof of Proposition 3(i).

Remark 1 Note that actually, in the above proof, the control u defined by $u(t) = -\frac{1}{\tau-t}\tilde{p}(t)$ for every $t \in (a_{\dagger} - a_2 + \varepsilon, \tau)$ is the feedback null control as defined by (8) (see Sect. 2) associated to the Lotka-McKendrick semigroup where the functions β and μ are defined by $\beta(a) := M$ and $\mu(a) := 0$ for almost every $a \in (0, a_2)$.

Proof of Proposition 3(ii) and (iii) The proof of Proposition 3(ii) follows the proof of Proposition 2(i), taking $a_b = 0$ and $a_1 > 0$, so we omit it. Moreover, the proof of Proposition 3(iii) is the same as the proof of Proposition 2(ii), Case 1.

Note that the proof of Theorem 3 follow closely the proofs of Theorems 1 and 2 (see Sects. 3 and 4), so we omit it.

6 An application of the linear controllability results: from a population to another one

In this section, we give an application of Propositions 1 and 3 to a linear population control problem. The aim of this section is, roughly speaking, to steer an initial population density to another population density in sufficiently large time, dealing with age restriction. More precisely, let τ be a positive constant and assume that \tilde{p} is solution to the system:

$$\begin{cases} \frac{\partial \tilde{p}}{\partial t}(t,a) + \frac{\partial \tilde{p}}{\partial a}(t,a) + \mu_0(t,a)\tilde{p}(t,a) = \chi_{[a_1,a_2]}(a)w(t,a), & (t,a) \in Q_\tau \\ \tilde{p}(t,0) = \int_0^{a_{\dagger}} \beta_0(t,a)\tilde{p}(t,a) \, \mathrm{d}a, & t \in (0,\tau) \\ \tilde{p}(0,a) = \tilde{p}_0(a), & a \in (0,a_{\dagger}), \end{cases}$$
(48)

where the functions μ_0 and β_0 are assumed to follow (L1)–(L4) (see Sect. 3), $w \in L^2(Q_\tau)$ is a nonnegative function and $\tilde{p}_0 \in L^2(0, a_{\dagger})$ is a nonnegative initial population density. We aim to find a control function $u \in L^2(Q_\tau)$ such that the corresponding unique solution p to (10) coincides with the solution \tilde{p} to (48) in time τ , while preserving the nonnegativity of p. Such a problem has been studied, for instance, in the pioneering work [6] and more recently in [8], assuming that the functions μ_0 , β_0 and w do not depend on the time, the control function u is supported in the age interval $(0, a_0)$ for some $a_0 < a_{\dagger}$ and \tilde{p} is a steady state of (10). The main novelty we bring here is that we may be able to steer the solution p of (10) to the solution \tilde{p} of (48) by means of a feedback control function u, without considering the above-mentioned restrictions. More precisely, we have

Proposition 4 With the above notations and assumptions, assume that at least one of the two following conditions is fulfilled.

Condition 1 There exists $a_b \in (0, a_{\dagger})$ such that (12) is satisfied, with $a_1 < a_b$ and $\tau > a_{\dagger} - a_2 + a_1$,

Condition 2 $a_1 = 0$ and $\tau > a_{\dagger} - a_2$.

Suppose that w and \tilde{p} are both nonnegative functions in $L^2(Q_{\tau})$, where \tilde{p} is the solution of (48). Then for every $p_0 \in L^2(0, a_{\dagger})$ with $p_0(a) \ge 0$ a.e. $a \in (0, a_{\dagger})$, there exists $u \in L^2(Q_{\tau})$ and a unique nonnegative solution p of (10) such that

$$p(\tau, a) = \tilde{p}(\tau, a)$$
 a.e. $a \in (0, a_{\dagger})$.

Proof Suppose, first, that the Condition 1 of Proposition 4 is fulfilled. For almost every $(t, a) \in Q_{\tau}$, we set

$$u(t, a) := w(t, a) - v(t, a)(p(t, a) - \tilde{p}(t, a)) \text{ a.e. } (t, a) \in Q_{\tau},$$
(49)

where v is defined by (13) [resp. by (21)] in the case where $a_2 \leq a_b$ (resp. in the case where $a_b < a_2$).

Denoting by $h := p - \tilde{p}$, using (10), (48) together with the above definition of the control function *u* given by (49), we get that *h* is solution of the following system:

$$\begin{cases} \frac{\partial h}{\partial t}(t,a) + \frac{\partial h}{\partial a}(t,a) \\ +\mu_0(t,a)h(t,a) = -\chi_{[a_1,a_2]}(a)v(t,a)h(t,a), \quad (t,a) \in Q_{\tau} \\ h(t,0) = \int_0^{a_{\dagger}} \beta_0(t,a)h(t,a) \, \mathrm{d}a, \qquad t \in (0,\tau) \\ h(0,a) = h_0(a), \qquad a \in (0,a_{\dagger}), \end{cases}$$
(50)

where $h_0(a) := p_0(a) - \tilde{p}_0(a)$ for almost every $a \in (0, a_{\dagger})$. From the proof of Proposition 1, we get that the solution h of (50) satisfies $h(\tau, \cdot) = 0$, with $vh \in L^2(Q_{\tau})$. It follows that we have $p(\tau, \cdot) = \tilde{p}(\tau, \cdot)$ and $u = w - vh \in L^2(Q_{\tau})$.

It remains to show that the unique solution p to (10) corresponding to u defined by (49) is nonnegative. Since u is defined by (49), we get that p is solution to

$$\begin{cases} \frac{\partial p}{\partial t}(t,a) + \frac{\partial p}{\partial a}(t,a) \\ +\tilde{\mu}(t,a)p(t,a) = \chi_{[a_1,a_2]}(a)(w(t,a) + v(t,a)\tilde{p}(t,a)), & (t,a) \in Q_{\tau} \\ p(t,0) = \int_{0}^{a_{\dagger}} \beta_{0}(t,a)p(t,a) \, \mathrm{d}a, & t \in (0,\tau) \\ p(0,a) = p_{0}(a), & a \in (0,a_{\dagger}), \end{cases}$$
(51)

where $\tilde{\mu}(t, a) := \mu_0(t, a) + \chi_{[a_1, a_2]}v(t, a)$ for almost every $(t, a) \in Q_\tau$. Noting that, for almost every $(t, a) \in Q_\tau$ we have

$$\chi_{[a_1,a_2]}(a)(w(t,a) + v(t,a)\tilde{p}(t,a)) \ge 0$$
 a.e. $(t,a) \in Q_{\tau}$

it follows from the comparison principle for linear age-structured population dynamics [see Theorem 4(i)] that the solution p of (51) satisfies $p(t, a) \ge 0$ for almost every $(t, a) \in Q_{\tau}$.

In the case where the Condition 2 is fulfilled, we may consider v as defined by (39) and follow the above proof to get the desired result.

7 Lack of null controllability for the Lotka-McKendrick equation with spatial diffusion and positivity constraints

In this section, we consider a linear controlled age-structured population model with spatial diffusion described by the following system:

$$\begin{split} \frac{\partial p}{\partial t}(t, a, x) &+ \frac{\partial p}{\partial a}(t, a, x) + \mu(t, a)p(t, a, x) - \Delta p(t, a, x) \\ &= \chi_{\omega}(x)u(t, a, x), & (t, a) \in Q_{\tau}, \ x \in \Omega \\ \frac{\partial p}{\partial \nu}(t, a, x) &= 0, & (t, a) \in Q_{\tau}, \ x \in \partial \Omega \\ p(t, 0, x) &= \int_{0}^{a_{\uparrow}} \beta(t, a)p(t, a, x) \, \mathrm{d}a, & t \in (0, \tau), \ x \in \Omega \\ p(0, a, x) &= p_{0}(a, x), & a \in (0, a_{\uparrow}), \ x \in \Omega. \end{split}$$
(52)

In the above equations:

- Ω ⊂ ℝ^N, N ≥ 1, denotes a smooth connected bounded domain and Δ is the Laplacian with respect to the variable x;
- $\frac{\partial}{\partial v}$ denotes the derivation operator in the direction of the unit outer normal to $\partial \Omega$. We thus have homogeneous Neumann boundary conditions; thus, the considered population is isolated from the exterior of Ω ;
- p(t, a, x) denotes the distribution density of the population at time *t*, of age *a* at spatial position $x \in \Omega$;
- p_0 denotes the initial population distribution;
- $a_{\dagger} \in (0, +\infty)$ is the maximal age of an individual;
- β and μ are nonnegative functions denoting respectively the birth and death rates;
- ω ⊂⊂ Ω is a nonempty open subset of Ω and χ_ω denotes the characteristic function of ω.

Let τ be a positive constant. We make the following classical assumptions on β and μ :

(D1)
$$\beta \in L^{\infty}(Q_{\tau}), \beta(t, a) \ge 0$$
 for almost every $(t, a) \in Q_{\tau}$,
(D2) $\mu \in L^{1}_{loc}([0, \tau] \times [0, a_{\dagger})), \mu(t, a) \ge 0$ almost every $(t, a) \in Q_{\tau}$.

From a controllability view point, system (52) has been extensively studied in the past decades. The particular case when the control acts in the whole space (the case corresponding to $\omega = \Omega$) was investigated by Aniţa (see [5], p 148). The case when the control acts in a spatial subdomain ω was firstly studied by Ainseba [1], where the author proves the null controllability of the above system (52), except for a small interval of ages near zero. The case when the control acts in a spatial subdomain ω and also only for small age classes was investigated by Ainseba and Aniţa [2], for initial data p_0 in a neighborhood of the target \tilde{p} . Related approximate and exact controllability issues have also been studied in Ainseba and Langlais [4], Ainseba and Iannelli, Traore [17], Kavian and Traore [12].

In a recent work, Hegoburu and Tucsnak [9] proved that, in the case where both functions β and μ do not depend on the time variable, the above system (52) is null

controllable in any time $\tau > 0$, in the sense that for any $p_0 \in L^2((0, a_{\dagger}) \times \Omega)$, there exists a control function $u \in L^{\infty}(0, \tau; L^2((0, a_{\dagger}) \times \Omega))$ such that the corresponding solution p of (52) satisfies

$$p(\tau, a, x) = 0$$
 a.e. $(a, x) \in (0, a_{\dagger}) \times \Omega$.

This result has been recently improved by Maity et al. [15], assuming that the young individuals are not able to reproduce before some age $a_b > 0$, where the control function u in system (52) has support in some interval of ages $[a_1, a_2]$, where $0 \leq a_1 < a_2 \leq a_{\dagger}$. In [15], the authors proved the null controllability result with this additional age restriction, provided that the control time τ is large enough, and the age a_1 is smaller than a_b .

The aim of this section is to prove that, in general, the solution p of the controlled system may not be positive in the whole time interval $[0, \tau]$. More precisely, we have:

Proposition 5 Let $\tau > 0$. With the above notations and assumptions, suppose that for almost every $t \in [0, \tau]$, we have that

$$\beta(t, \cdot)$$
 is positive on a subset of positive measure of $(0, a_{\dagger})$, (53)

and denote by

$$E_t := \{a \in (0, a_{\dagger}); \ \beta(t, a) \neq 0\}$$
 and $E_t^+ := \sup E_t.$ (54)

Suppose that

$$m := \inf \left\{ E_t^+, \ t \in [0, \tau] \right\} > 0.$$
(55)

Then there exists $p_0 \in L^2((0, a_{\dagger}) \times \Omega)$ with $p_0(a, x) \ge 0$ for almost every $(a, x) \in (0, a_{\dagger}) \times \Omega$ such that there does not exists a control function $u \in L^2((0, \tau); L^2((0, a_{\dagger}) \times \Omega))$ such that

$$p(\tau, a, x) = 0 \quad a.e. \ (a, x) \in (0, a_{\dagger}) \times \Omega,$$

together with

$$p(t, a, x) \ge 0$$
 a.e. $(t, a, x) \in Q_{\tau} \times \Omega$.

In other words, for some well chosen nonnegative initial condition $p_0 \in L^2((0, a_{\dagger}) \times \Omega)$, it is not possible to eradicate the whole population in time τ while keeping the nonnegativity of the state trajectory.

Proof Let $\tau > 0$, and assume that (53) and (55) are satisfied. Denote by ω^c the complementary of ω in Ω . Let $h_0 \in L^2(\Omega)$ be a nonnegative function such that for almost every $x \in \omega^c$ we have

$$h_0(x) > 0 \quad \text{a.e. } x \in \omega^c. \tag{56}$$

For almost every $(a, x) \in (0, a_{\dagger}) \times \Omega$, we set

$$p_0(a, x) := h_0(x). \tag{57}$$

It is clear that the initial condition p_0 defined by (57) belongs to $L^2((0, a_{\dagger}) \times \Omega)$ and is a nonnegative function. Let $\tau > 0$. Suppose that there exists a control function $u \in L^2((0, \tau); L^2((0, a_{\dagger}) \times \Omega))$ such that the two following conditions are fulfilled:

$$p(\tau, a, x) = 0$$
 a.e. $(a, x) \in (0, a_{\dagger}) \times \Omega$,

and

$$p(t, a, x) \ge 0$$
 a.e. $(t, a, x) \in Q_{\tau} \times \Omega$,

where p is the solution of (52).

In the subdomain $Q_{\tau} \times \omega^c$, given (52) and (57), the function p satisfies

$$\begin{cases} \frac{\partial p}{\partial t}(t, a, x) + \frac{\partial p}{\partial a}(t, a, x) \\ +\mu(t, a)p(t, a, x) - \Delta p(t, a, x) = 0, \quad (t, a) \in Q_{\tau}, \ x \in \omega^{c} \\ \frac{\partial p}{\partial \nu}(t, a, x) = 0, \quad (t, a) \in Q_{\tau}, \ x \in \partial \Omega \\ p(t, 0, x) = \int_{0}^{a_{\dagger}} \beta(t, a)p(t, a, x) \, \mathrm{d}a, \quad t \in (0, \tau), \ x \in \omega^{c} \\ p(0, a, x) = h_{0}(x), \quad a \in (0, a_{\dagger}), \ x \in \omega^{c}. \end{cases}$$

By the comparison principle, for almost every $(t, a, x) \in Q_{\tau} \times \omega^{c}$, we have that

$$p(t, a, x) \ge \tilde{p}(t, a, x) \ge 0, \tag{58}$$

where the function \tilde{p} satisfies the following equation:

$$\begin{cases} \frac{\partial \tilde{p}}{\partial t}(t, a, x) + \frac{\partial \tilde{p}}{\partial a}(t, a, x) \\ +\mu(t, a)\tilde{p}(t, a, x) - \Delta \tilde{p}(t, a, x) = 0, \quad (t, a) \in Q_{\tau}, \ x \in \omega^{c} \\ \tilde{p}(t, a, x) = 0, \quad (t, a) \in Q_{\tau}, \ x \in \partial \omega \\ \frac{\partial \tilde{p}}{\partial \nu}(t, a, x) = 0, \quad (t, a) \in Q_{\tau}, \ x \in \partial \Omega \\ \tilde{p}(t, 0, x) = \int_{0}^{a_{\dagger}} \beta(t, a)\tilde{p}(t, a, x) \, \mathrm{d}a, \quad t \in (0, \tau), \ x \in \omega^{c} \\ \tilde{p}(0, a, x) = h_{0}(x), \quad a \in (0, a_{\dagger}), \ x \in \omega^{c}. \end{cases}$$
(59)

Note that the above system (59) is separable, in the sense that the solution \tilde{p} to (59) writes as

$$\tilde{p}(t, a, x) = g(t, a)h(t, x) \quad \text{a.e.} \ (t, a, x) \in Q_{\tau} \times \omega^c, \tag{60}$$

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where the functions $g \in C([0, \tau] \times [0, a_{\dagger}] \setminus S)$ ($S = \{(t, a) \in [0, \tau] \times [0, a_{\dagger}]; t = a\}$) and $h \in C([0, \tau]; L^2(\Omega))$ are respectively solutions to

$$\begin{cases} \frac{\partial g}{\partial t}(t,a) + \frac{\partial g}{\partial a}(t,a) + \mu(t,a)g(t,a) = 0, & (t,a) \in Q_{\tau} \\ g(t,0) = \int_{0}^{a_{\dagger}} \beta(t,a)g(t,a) \, \mathrm{d}a, & t \in (0,\tau) \\ g(0,a) = 1, & a \in (0,a_{\dagger}), \end{cases}$$
(61)

and

$$\begin{cases} \frac{\partial h}{\partial t}(t, x) - \Delta h(t, x) = 0, & t \in (0, \tau), \ x \in \omega^c \\ h(t, x) = 0, & t \in (0, \tau), \ x \in \partial \omega \\ \frac{\partial h}{\partial \nu}(t, x) = 0, & t \in (0, \tau), \ x \in \partial \Omega \\ h(0, x) = h_0(x), & x \in \omega^c. \end{cases}$$

Given assumptions (53) and (55), let us prove that, for every $\tau > 0$, we have that $g(\tau, a) > 0$ on a subset of positive measure of $(0, a_{\dagger})$. To this aim, integrating along the characteristics the equation satisfied by g, for almost every $(t, a) \in Q_{\tau}$ we have

$$g(t,a) = \begin{cases} \exp\left(-\int_0^t \mu(s, a-t+s) \,\mathrm{d}s\right), & a \ge t, \\ \exp\left(-\int_0^a \mu(t-a+s, s) \,\mathrm{d}s\right) B(t-a), & a < t, \end{cases}$$
(62)

where

$$B(t) = \int_0^{a_{\dagger}} \beta(t, a) g(t, a) \, \mathrm{d}a \quad \text{a.e. } t \in (0, \tau).$$

Suppose that there exists $\tau > 0$ such that for almost every $a \in (0, a_{\dagger})$ we have g(t, a) = 0, where g satisfies (61). Given (62), necessarily, we have $\tau \ge a_{\dagger}$. Assuming that $g(\tau, a) = 0$ for almost every $a \in (0, a_{\dagger})$ for some $\tau \ge a_{\dagger}$, from (62) we have

$$B(s) = 0$$
 a.e. $s \in (\tau - a_{\dagger}, \tau)$,

so that, in particular, we have

$$B(\tau - a_{\dagger}) = \int_{E_{\tau - a_{\dagger}}} \beta(\tau - a_{\dagger}, a)g(\tau - a_{\dagger}, a) \,\mathrm{d}a = 0, \tag{63}$$

where $E_{\tau-a_{\dagger}}$ is defined by (54). Since $\beta(\tau - a_{\dagger}, a) > 0$ for almost every $a \in E_{\tau-a_{\dagger}}$ and *p* is assumed to be positive, from (63) we get that for almost every $a \in E_{\tau-a_{\dagger}}$, we have

$$g(\tau - a_{\dagger}, a) = 0$$
 a.e. $a \in E_{\tau - a_{\dagger}}$. (64)

Denote by $E^{(1)} := E^+_{\tau - a_{\dagger}}$, where $E^+_{\tau - a_{\dagger}}$ is defined by (54). From (64), we have

$$g(\tau - a_{\dagger}, E^{(1)}) = 0.$$
(65)

If $E^{(1)} \ge \tau - a_{\dagger}$, the above condition (65) gives a contradiction with (62). Otherwise, we have $E^{(1)} < \tau - a_{\dagger}$ and integrating along the characteristics, we have

$$g(\tau - a_{\dagger} - E^{(1)}, 0) = 0.$$

Denote by $E^{(2)} = E^+_{\tau - a_{\tau} - E^{(1)}}$. Following the previous arguments, we may deduce from the renewal law that we have

$$g(\tau - a_{\dagger} - E^{(1)}, E^{(2)}) = 0.$$
 (66)

If $E^{(2)} \ge \tau - a_{\dagger} - E^{(1)}$, the above condition (66) gives a contradiction with (62). Otherwise, we have $E^{(2)} < \tau - a_{\dagger} - E^{(1)}$ and integrating along the characteristics, we have

$$g(\tau - a_{\dagger} - E^{(1)} - E^{(2)}, 0) = 0.$$

Define, recursively, the sequence $(E^{(n)})_{n \ge 1}$ by $E^{(1)} := E^+_{\tau - a_{\dagger}}$ and $E^{(n+1)} := E^+_{\tau - a_{\dagger} - E^{(1)} - \dots - E^{(n)}}$. Following the above procedure, we way obtain by induction that for every $n \ge 1$ such that $\tau - a_{\dagger} - E^{(1)} - \dots - E^{(n)} \ge 0$, we have

$$g(\tau - a_{\dagger} - E^{(1)} - \dots - E^{(n)}, E^{(n+1)}) = 0.$$

Recall that there exists m > 0 such that, for almost every $t \in [0, \tau]$ we have $E_t^+ \ge m > 0$. Thus, denote by *l* the first integer such that

$$\tau - a_{\dagger} - E^{(1)} - \dots - E^{(l)} - E^{(l+1)} \leq 0.$$

Then, the condition

$$g(\tau - a_{\dagger} - E^{(1)} - \dots - E^{(l)}, E^{(l+1)}) = 0,$$

gives a contradiction with (62), since $E^{(l+1)} \ge \tau - a_{\dagger} - E^{(1)} - \cdots - E^{(l)}$. It follows that $g(\tau, \cdot)$ cannot be the null function. By (60) we have that $h(\tau, x) = 0$ for almost every $x \in \omega^c$. From the backward uniqueness for the parabolic equations, we have $h_0(x) = 0$ for almost every $x \in \omega^c$. This is a contradiction with (56).

Remark 2 Using the comparison results for age-structured population dynamics with spatial diffusion and feedback controls of harvesting type, it is possible to obtain null-controllability results when the control function u acts in the whole habitat Ω , for any $t \in (0, \tau)$ but only for some age subinterval.

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