

CONTROLLABILITY WITH POSITIVITY CONSTRAINTS OF THE LOTKA–MCKENDRICK SYSTEM*

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Abstract. This work considers the linear Lotka–McKendrick system from population dynamics with control active on individuals in a prescribed age range. The main results assert that, given τ large enough (but possibly smaller than the life expectancy), there exist controls driving the system to any equilibrium state or any uncontrolled trajectory in time τ . Moreover, we show that if the initial and final states are positive, then the constructed controls preserve the positivity of the population density on the whole time interval $[0, \tau]$. The method is a direct one, in the spirit of some early works on the controllability of hyperbolic systems in one space dimension. Finally, we apply our method to a nonlinear infection-age model.

Key words. population dynamics, controllability, positivity, positive trajectories

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1. Introduction. The purpose of this work is the study of the controllability properties of an infinite-dimensional dynamical system coming from population dynamics. More precisely, we revisit from a control theoretical viewpoint the linear Lotka–McKendrick model. We aim, in particular, to answer some open questions raised in Barbu, Iannelli, and Martcheva [8]. More precisely, let $p(t, a)$ be the distribution of individuals of age $a \geq 0$ at time $t \geq 0$. Let a_+ be the life expectancy of an individual, and let τ be a positive constant. Let $\beta(a) \geq 0$ be the natural fertility rate and $\mu(a) \geq 0$ the natural death rate of individuals of age a . The system we consider, already studied from a control theoretic viewpoint in [8], is described by the equations

$$(1.1) \quad \begin{cases} \frac{\partial p}{\partial t}(t, a) + \frac{\partial p}{\partial a}(t, a) + \mu(a)p(t, a) \\ \quad = m(a)u(t, a), & (t, a) \in Q_\tau, \\ p(t, 0) = \int_0^{a_+} \beta(a)p(t, a) da, & t \in (0, \tau), \\ p(0, a) = p_0(a), & a \in (0, a_+), \end{cases}$$

where u is a control function, $m = \mathbb{1}_{[0, a_0]}$ is the characteristic function of the interval $(0, a_0)$ (where $0 < a_0 < a_+$), p_0 is the initial population density, and $Q_\tau = (0, \tau) \times (0, a_+)$.

In [8] the main result asserts controllability of the above system to any quasi steady state by a distributed control, except for a small interval of ages near zero. The main open questions raised in the above-mentioned reference are the possibility of controlling the whole range of ages and designing controls which preserve the positivity of the state trajectories. We give a positive answer to both questions above, in the particular case when the age of individuals able to reproduce is bounded away from zero.

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To state our main results, we first recall the standard assumptions, used in particular in [8], on the functions μ and β :

(H1) $\beta \in L^\infty(0, a_+)$, $\beta(a) \geq 0$ for almost every $a \in (0, a_+)$.

(H2) $\mu \in L^1[0, a^*]$ for every $a^* \in (0, a_+)$, $\mu(a) \geq 0$ for almost every $a \in (0, a_+)$.

(H3) $\int_0^{a_+} \mu(a) da = +\infty$.

The controllability result [8] asserts that if $\delta \in (0, a_0]$ and if \tilde{p} is a time-independent function which satisfies the equations

$$(1.2) \quad \begin{cases} \frac{\partial \tilde{p}}{\partial a}(a) + \mu(a)\tilde{p}(a) = m(a)w(a) & (a \in (0, a_+)), \\ \tilde{p}(0) = \int_0^{a_+} \beta(a)\tilde{p}(a) da \end{cases}$$

for some function $w \in L^2[0, a_+]$, then for every $p_0 \in L^2[0, a_+]$ and every time

$$(1.3) \quad a_+ \leq \tau < a_+ + \delta,$$

there exists $u \in L^2([0, \tau]; L^2[0, a_+])$ such that

$$p(\tau, a) = \tilde{p}(a) \quad (a \in [\delta, a_+]).$$

In other words, the above result from [8] asserts that we can control in a time τ satisfying (1.3) the population of individuals of age at least equal to some $\delta > 0$. Our main result asserts that this restriction is not necessary, provided that we assume that the age of individuals able to reproduce is bounded away from zero. More precisely, we have the following.

THEOREM 1.1. *With the above notation and assumptions, suppose that there exists $a_b \in (0, a_+)$ such that*

$$(1.4) \quad \beta(a) = 0 \quad (a \in (0, a_b) \text{ a.e.}).$$

Then for every $\tau > a_+ - a_0$, for every $p_0 \in L^2([0, a_+])$ with $p_0(a) \geq 0$ for almost every $a \in (0, a_+)$, and for any positive function \tilde{p} satisfying (1.2), there exists $u \in L^2([0, \tau]; L^2[0, a_+])$ such that the solution p of (1.1) satisfies

$$p(\tau, a) = \tilde{p}(a) \quad (a \in (0, a_+) \text{ a.e.}),$$

and

$$p(t, a) \geq 0 \text{ a.e. } (t, a) \in Q_\tau.$$

For an overview on age-structured population dynamics models, we refer the reader to Webb [26], Iannelli [13], Kunisch, Schappacher, and Webb [18] and references therein. For related work on size-structured population dynamics, see Ackleh and Ito [1] and Ito, Kappel, and Peichl [14].

The literature devoted to the associated control problems is less abundant, but several important results and methods are available. For optimal control problems (namely applied to human population) we refer the reader to Song and Yu [22]. The null controllability of the age-dependent population dynamics (1.1) with spatial dependence in the particular case when the control acts for all ages a (the case corresponding to $a_0 = a_+$) was investigated by Anița (see [6, p. 148]). The case when

the control acts in a spatial subdomain ω and only for small age classes was investigated by Ainseba and Anița [3] for initial data p_0 in a neighborhood of the target \tilde{p} . Related approximate and exact controllability issues have also been studied in Ainseba [2], Ainseba and Langlais [5], Ainseba and Iannelli [4], Traore [24], and Kaviani and Traore [17]. Using a direct approach, the approximate controllability by birth or boundary control was studied in Yu, Guo, and Zhu [27].

The remaining part of this work is organized as follows. In section 2 we recall some basic results on the Lotka–McKendrick semigroup. Section 3 is devoted to the study of the reachable space of (1.1). In section 4 we investigate some controllability properties of system (1.1). In particular, we give the proof of Theorem 1.1. In section 5 we discuss a singular perturbation problem. More precisely we show that if the age interval on which our control is active shrinks to $\{0\}$, the system (including controls and state trajectories) converges to a direct birth controlled system (which can be seen as an “impulse” control). In section 6 we show how our results apply to a nonlinear system arising in an infection-age model. We close with an appendix showing that the considered system is still null controllable in an L^1 setting.

2. Some background on the Lotka–McKendrick semigroup. In this section we recall, with no claim of originality, the formulation of equations (1.1) using semigroup theory.

Denote $X = L^2[0, a_+]$ and consider the operator $A : \mathcal{D}(A) \rightarrow X$ defined by

$$(2.1) \quad \begin{aligned} \mathcal{D}(A) &= \left\{ \varphi \in L^2[0, a_+] \mid \varphi(0) = \int_0^{a_+} \beta(a)\varphi(a) da; -\frac{d\varphi}{da} - \mu\varphi \in L^2[0, a_+] \right\}, \\ A\varphi &= -\frac{d\varphi}{da} - \mu\varphi \quad (\varphi \in \mathcal{D}(A)). \end{aligned}$$

It is well known (see, for instance, Song et al. [23] or Kappel and Zhang [16]) that A generates a C^0 semigroup of linear operators in X which we denote by $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$. This allows us to define the concept of a (mild) solution of (1.1) in the following standard way: We say that p is a mild solution of (1.1) if

$$(2.2) \quad p(t, \cdot) = \mathbb{T}_t p_0 + \Phi_t u \quad (t \geq 0, \quad u \in L^2([0, \infty); X)),$$

where the control operator $B \in \mathcal{L}(X)$ is defined by

$$Bu = mu \quad (u \in X),$$

and where

$$(2.3) \quad \Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma) d\sigma \quad (t \geq 0, \quad u \in L^2([0, \infty); X)).$$

It is well known (see, for instance, [13] or [26]) that the semigroup \mathbb{T} satisfies

$$(2.4) \quad (\mathbb{T}_t f)(a) = \begin{cases} \frac{\pi(a)}{\pi(a-t)} f(a-t) & \text{if } t \leq a, \\ \pi(a)b(t-a) & \text{if } t > a, \end{cases}$$

where $\pi(a) = e^{-\int_0^a \mu(\sigma) d\sigma}$ is the probability of survival of an individual from age 0 to a and $b(t) = (\mathbb{T}_t f)(0) = \int_0^{a_+} \beta(a)(\mathbb{T}_t f)(a) da$ is the total birth rate function. Moreover,

according, for instance, to [13, p. 12], b is the unique continuous function satisfying the integral Volterra equation of the second kind:

$$(2.5) \quad b(t) = \int_0^{\min(t, a_+)} \beta(t-s)\pi(t-s)b(s) \, ds + \int_{\min(t, a_+)}^{a_+} \beta(a) \frac{\pi(a)}{\pi(a-t)} f(a-t) \, da.$$

The state trajectory p of the controlled system (1.1) is given (see, for instance, [8]) by

$$(2.6) \quad p(t, a) := \begin{cases} \frac{\pi(a)}{\pi(a-t)} p_0(a-t) + c(t, a), & t \leq a, \\ \pi(a) \tilde{b}(t-a) + c(t, a), & a < t, \end{cases}$$

where

$$(2.7) \quad c(t, a) := \begin{cases} \int_{a-t}^a \frac{\pi(a)}{\pi(s)} m(s) u(s - (a-t), s) \, ds, & t \leq a, \\ \int_0^a \frac{\pi(a)}{\pi(s)} m(s) u(s + (t-a), s) \, ds, & a < t, \end{cases}$$

where $\tilde{b}(t) = \int_0^{a_+} \beta(a) p(t, a) \, da$ is the flux of newly born individuals for the controlled system. It is easy to check that \tilde{b} is the unique continuous function satisfying the Volterra integral equation

$$(2.8) \quad \begin{aligned} \tilde{b}(t) = & \int_0^{\min(t, a_+)} \beta(t-s) \left[\pi(t-s) \tilde{b}(s) + c(t, t-s) \right] \, ds \\ & + \int_{\min(t, a_+)}^{a_+} \beta(a) \left[\frac{\pi(a)}{\pi(a-t)} p_0(a-t) + c(t, a) \right] \, da. \end{aligned}$$

Remark 2.1. The control u constructed in the following section will be of the form

$$u(t, a) = v(t-a) \quad (a \in [0, a_+], t \in [0, \tau]),$$

where $v : [-a_+, \tau] \rightarrow \mathbb{R}$. In this case (2.6) becomes

$$(2.9) \quad p(t, a) = \begin{cases} \frac{\pi(a)}{\pi(a-t)} p_0(a-t) + v(t-a) \int_{a-t}^a \frac{\pi(a)}{\pi(s)} m(s) \, ds, & a \geq t, \\ \pi(a) \tilde{b}(t-a) + v(t-a) \int_0^a \frac{\pi(a)}{\pi(s)} m(s) \, ds, & a < t. \end{cases}$$

Remark 2.2. If the state space $X = L^2[0, a_+]$ is substituted by $X = L^1[0, a_+]$, the operator $A : \mathcal{D}(A) \rightarrow X$ defined by

$$(2.10) \quad \begin{aligned} \mathcal{D}(A) = & \left\{ \varphi \in L^1[0, a_+] \mid \varphi(0) = \int_0^{a_+} \beta(a) \varphi(a) \, da; -\frac{d\varphi}{da} - \mu\varphi \in L^1[0, a_+] \right\}, \\ A\varphi = & -\frac{d\varphi}{da} - \mu\varphi \quad (\varphi \in \mathcal{D}(A)) \end{aligned}$$

generates a C^0 semigroup of linear operators in X denoted by $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$ (see, for instance, Banks and Kappel [7]). This allows us to define the concept of a (mild)

solution of (1.1) in the same way as above: We say that p is a mild solution of (1.1) if

$$(2.11) \quad p(t, \cdot) = \mathbb{T}_t p_0 + \Phi_t u \quad (t \geq 0, \quad u \in L^1([0, \infty); X)),$$

where the control operator $B \in \mathcal{L}(X)$ is defined by

$$Bu = mu \quad (u \in X),$$

and where

$$(2.12) \quad \Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma) \, d\sigma \quad (t \geq 0, \quad u \in L^1([0, \infty); X)).$$

The semigroup \mathbb{T} satisfies the same formulas (2.4) as in the L^2 setting.

3. Study of the reachable space. In this section, we study the reachable space of (1.1) by means of controls $u \in L^2([0, \tau]; L^2[0, a_\dagger])$. In other words, we study the space $\text{Ran } \Phi_\tau$, for every $\tau > 0$, where the operator Φ_τ has been defined in (2.3). In order to obtain controllability results, we focus on the case $\tau > a_\dagger - a_0$ since in the opposite case, as shown below, we have neither approximate nor null controllability. For the sake of completeness, we recall that the property of *approximate controllability* in some time $\tau > 0$ means that $\text{Ran } \Phi_\tau$ is dense in X and that the *null controllability* in time τ means that $\text{Ran } \Phi_\tau \supset \text{Ran } \mathbb{T}_\tau$ (we refer the reader, for instance, to [25, Chapter 11] for a detailed description of these concepts).

PROPOSITION 3.1. *Suppose that $\tau < a_\dagger - a_0$. Then system (1.1) is neither approximately controllable nor null controllable in time τ .*

Proof. Let $\tau \in]0, a_\dagger - a_0[$. Then for every $u \in L^2([0, \tau]; L^2[0, a_\dagger])$ and for almost every $a \in (a_0 + \tau, a_\dagger)$, we have from (2.6) that

$$(3.1) \quad (\Phi_\tau u)(a) = \int_{a-\tau}^a \frac{\pi(a)}{\pi(s)} m(s) u(s + (\tau - a), s) \, ds = 0,$$

since $m(s) = 0$ if $s \geq a - \tau > a_0$. This clearly implies that system (1.1) is not approximately controllable in time τ .

On the other hand, (2.4) implies that there exists $p_0 \in X$ such that $(\mathbb{T}_\tau p_0)(a) \neq 0$ on $[a_0 + \tau, a_\dagger]$. Consequently, $\text{Ran } \Phi_\tau \not\supset \text{Ran } \mathbb{T}_\tau$, so that (1.1) is not null controllable in time τ . \square

The following is the main result.

THEOREM 3.2. *Under the assumptions of Theorem 1.1, define*

$$(3.2) \quad \mathcal{R} := \left\{ \tilde{p} \in X \mid a \mapsto \frac{\tilde{p}(a)}{\pi(a)} \in X, \quad a \mapsto \frac{\frac{\tilde{p}(a)}{\pi(a)} - \int_0^{a_\dagger - a} \beta(x) \frac{\pi(x)}{\pi(a+x)} \tilde{p}(a+x) \, dx}{\int_0^a \frac{m(x)}{\pi(x)} \, dx} \in X \right\}.$$

Then for every $\tau > a_\dagger - a_0$ we have $\mathcal{R} \subset \text{Ran } \Phi_\tau$.

Proof. We can assume, without loss of generality, that $a_0 \leq a_b$ (otherwise we simply control the system for smaller ages). Let $\tau \in (a_\dagger - a_0, a_\dagger)$.

We define the sets $\{P_\tau^j\}_{j \geq -1}$ by

$$P_\tau^{-1} = \{(t, a) \in (0, +\infty)^2 : \tau - a_\dagger \leq t - a \leq 0\}$$

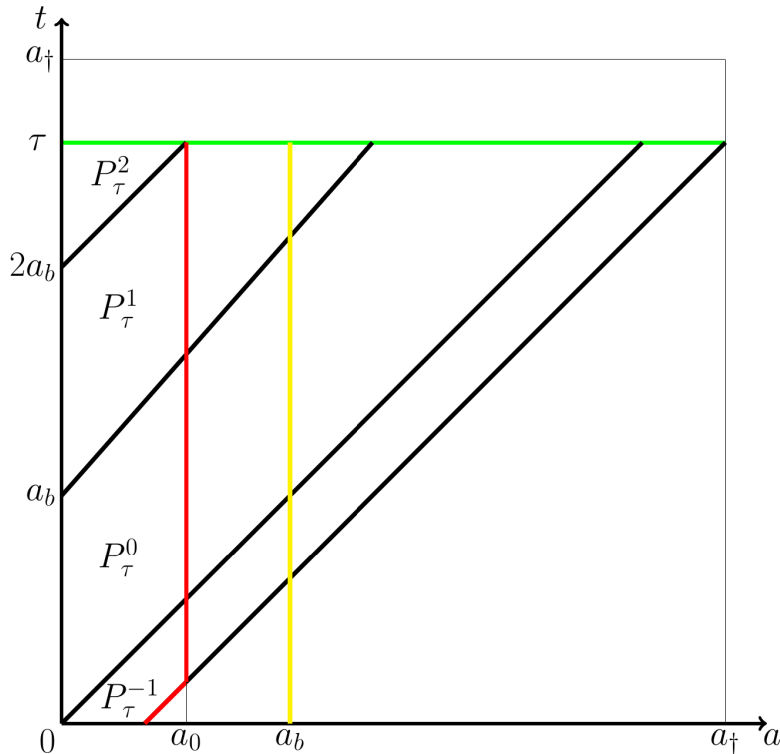


FIG. 1. In this figure the control possibly acts in the subset of $Q_\tau = [0, \tau] \times [0, a_+]$ delimited by the line $a = 0$ and the red curve. The domain Q_τ is decomposed into subregions $P_\tau^{-1}, P_\tau^0, P_\tau^1, P_\tau^2, \dots$, which are delimited, respectively, by the characteristics lines $t - a = \tau - a_+, t - a = 0, t - a = a_b, t - a = 2a_b, \dots$. For each subregion P_τ^j (for $j \geq 0$) the values of $\bar{b}(t) = \int_0^{a_+} \beta(a)p(t, a) da$ for $t \in [ja_b, (j+1)a_b]$ depend only on the control in the previous subregions $P_\tau^{-1}, \dots, P_\tau^{j-1}$. This will allow us, using an inductive method, to derive an explicit formula of the control functions in terms of the targeted distribution of population. (Color available online.)

and for $j \geq 0$

$$P_\tau^j = \{(t, a) \in (0, +\infty)^2 : ja_b < t - a \leq (j+1)a_b\}.$$

The control region and the sets P_τ^j $j \geq -1$ are pictured in Figure 1.

Given $\tilde{p} \in \mathcal{R}$, we look for $v \in L^2[-a_+, \tau]$ such that, setting $u(t, a) = v(t - a)$ for every $t \in [0, \tau]$ and $a \in [0, a_+]$, we have

$$(3.3) \quad \Phi_\tau u(a) = \tilde{p}(a) \quad (a \in [0, a_+] \text{ a.e.})$$

Case $t - a < \tau - a_+$. We first set $v(s) = 0$ for $s \in [-a_+, \tau - a_+]$ so that, according to (2.9), we have

$$(3.4) \quad (\Phi_t u)(a) = 0 \quad (t - a < \tau - a_+).$$

Case $\tau - a_+ \leq t - a \leq 0$. By using again (2.9), it follows that

$$(\Phi_\tau u)(a) = v(\tau - a) \int_{a-\tau}^a \frac{\pi(a)}{\pi(s)} m(s) ds \quad (a \in [\tau, a_+] \text{ a.e.}).$$

Consequently, (3.3) is satisfied for $a \in [\tau, a_+]$ iff

$$(3.5) \quad v(s) = \frac{\tilde{p}(\tau-s)}{\int_{-s}^{\tau-s} \frac{m(x)}{\pi(x)} dx} \quad (s \in [\tau - a_+, 0] \text{ a.e.}).$$

Note that since $\tau > a_+ - a_0$, $s \in [\tau - a_+, 0]$ implies that $-s < a_0$; therefore for every $s \in [\tau - a_+, 0]$ we have

$$\int_{-s}^{\tau-s} \frac{m(x)}{\pi(x)} dx > 0.$$

Moreover, by assumption we have $a \mapsto \frac{\tilde{p}(a)}{\pi(a)} \in L^2[0, a_+]$, and hence

$$v \in L^2[\tau - a_+, 0].$$

Case 0 < t - a < a_b. According to (2.9) and (3.5), for almost every $(t, a) \in P_\tau^{-1}$ we have

$$(3.6) \quad \begin{aligned} (\Phi_t u)(a) &= v(t-a) \int_{a-t}^a \frac{\pi(a)}{\pi(s)} m(s) ds \\ &= \pi(a) \frac{\int_{a-t}^a \frac{m(s)}{\pi(s)} ds}{\int_{a-t}^{\tau+(a-t)} \frac{m(s)}{\pi(s)} ds} \frac{\tilde{p}(\tau+(a-t))}{\pi(\tau+(a-t))}. \end{aligned}$$

To determine v on $[0, a_b]$ we note that from (2.9) it follows that

$$(3.7) \quad (\Phi_\tau u)(a) = \pi(a) \tilde{b}(\tau-a) + v(\tau-a) \int_0^a \frac{\pi(a)}{\pi(s)} m(s) ds \quad (a \in [\tau - a_b, \tau] \text{ a.e.}),$$

where

$$(3.8) \quad \tilde{b}(t) = \int_{a_b}^{a_+} \beta(a) (\Phi_t u)(a) da \quad (t \in [0, a_b]).$$

To determine \tilde{b} , after some simple calculations and using (3.4), (3.6), and (3.8), it follows that

$$(3.9) \quad \tilde{b}(t) = \int_{a_b}^{t+a_+-\tau} \beta(a) \pi(a) \frac{\tilde{p}(\tau+(a-t))}{\pi(\tau+(a-t))} da \quad (t \in [0, a_b]).$$

Inserting (3.9) into (3.7), it follows that

$$(3.10) \quad \begin{aligned} (\Phi_\tau u)(a) &= \pi(a) \int_{a_b}^{a_+-a} \beta(x) \pi(x) \frac{\tilde{p}(a+x)}{\pi(a+x)} dx \\ &\quad + v(\tau-a) \int_0^a \frac{\pi(a)}{\pi(s)} m(s) ds \quad (a \in [\tau - a_b, \tau] \text{ a.e.}). \end{aligned}$$

Thus, (3.3) is satisfied for $a \in [\tau - a_b, \tau]$ iff

$$(3.11) \quad v(s) = \frac{\tilde{p}(\tau-s) - \pi(\tau-s) \chi(s)}{\pi(\tau-s) \int_0^{\tau-s} \frac{m(x)}{\pi(x)} dx} \quad (s \in [0, a_b] \text{ a.e.}),$$

where

$$(3.12) \quad \chi(s) := \int_{a_b}^{s+a_+-\tau} \beta(x) \pi(x) \frac{\tilde{p}(\tau-(s-x))}{\pi(\tau-(s-x))} dx \quad (s \in [0, \tau]).$$

Since $\tilde{p} \in L^2[0, a_+]$, we clearly have that $v \in L^2[0, a_b]$.

Case $ja_b < t - a < (j - 1)a_b$ for some integer $j \geq 1$. The above calculations suggest, as a possible control, driving the zero initial state to $u(t, a) = v(t - a)$, where

$$(3.13) \quad v(s) = \begin{cases} 0 & \text{for } s \in [-a_\dagger, \tau - a_\dagger], \\ \frac{\tilde{p}(\tau-s)}{\pi(\tau-s)} & \text{for } s \in [\tau - a_\dagger, 0], \\ \frac{\tilde{p}(\tau-s) - \pi(\tau-s)\chi(s)}{\pi(\tau-s) \int_0^{\tau-s} \frac{m(x)}{\pi(x)} dx} & \text{for } s \in [0, \tau], \end{cases}$$

where $\chi(s)$ is defined by (3.12).

To prove this fact we first note that v defined above is in $L^2[-a_\dagger, \tau]$ due to the fact that $\tilde{p} \in \mathcal{R}$. Moreover, we have already seen that (3.13) gives

$$(\Phi_\tau u)(a) = \tilde{p}(a) \quad (a \in [\tau - a_b, a_\dagger] \text{ a.e.}).$$

We next show by induction over $j \geq 0$ the following properties, which we denote by (\mathcal{P}_j) :

- $\tilde{b}(t) = \chi(t)$ for every $t \in [ja_b, (j + 1)a_b]$.
- $(\Phi_\tau u)(a) = \tilde{p}(a)$ for a.e. $a \in (\tau - (j + 1)a_b, \tau - ja_b)$.

We have seen above that (\mathcal{P}_0) is true.

Let $j \geq 0$ be a fixed integer, and suppose that (\mathcal{P}_i) holds for every $i \in \{0, \dots, j\}$. Then, for every $i \in \{0, \dots, j\}$ and for almost every $(t, a) \in P_\tau^i$, from (2.9) and (3.13), we have

$$(3.14) \quad (\Phi_t u)(a) = \pi(a)\chi(t - a) \left[1 - \frac{\int_0^a \frac{m(s)}{\pi(s)} ds}{\int_0^{\tau-(t-a)} \frac{m(s)}{\pi(s)} ds} \right] + \pi(a) \frac{\int_0^a \frac{m(x)}{\pi(x)} dx}{\int_0^{\tau-(t-a)} \frac{m(x)}{\pi(x)} dx} \frac{\tilde{p}(\tau - (t - a))}{\pi(\tau - (t - a))}.$$

Combining (3.4), (3.6), and the above formula (3.14), we are in position to compute $\tilde{b}(t)$ for $t \in [(j + 1)a_b, (j + 2)a_b]$. Indeed, since $\tilde{b}(t) = \int_{a_b}^{a_\dagger} \beta(a)(\Phi_t u)(a) da$ for every $t \in [(j + 1)a_b, (j + 2)a_b]$, we can use (3.4) to obtain that

$$(3.15) \quad \tilde{b}(t) = \int_{a_b}^t \beta(a)(\Phi_t u)(a) da + \int_t^{t+a_\dagger-\tau} \beta(a)(\Phi_t u)(a) da \quad (t \in [(j + 1)a_b, (j + 2)a_b]).$$

First according to (3.14), the first integral in the above formula (3.15) writes

$$\int_{a_b}^t \beta(a)(\Phi_t u)(a) da = \int_{a_b}^t \beta(a) \left(\pi(a)\chi(t - a) \left[1 - \frac{\int_0^a \frac{m(s)}{\pi(s)} ds}{\int_0^{\tau-(t-a)} \frac{m(s)}{\pi(s)} ds} \right] + \pi(a) \left[\frac{\int_0^a \frac{m(x)}{\pi(x)} dx}{\int_0^{\tau-(t-a)} \frac{m(x)}{\pi(x)} dx} \right] \frac{\tilde{p}(\tau - (t - a))}{\pi(\tau - (t - a))} \right) da.$$

We note that the term contained in the first bracket of the above formula involves only values of a which are larger than a_0 , so it vanishes. Moreover the term in the

second bracket of the last formula is clearly equal to one. Consequently,

$$(3.16) \quad \int_{a_b}^t \beta(a)(\Phi_t u)(a) da = \int_{a_b}^t \beta(a)\pi(a) \frac{\tilde{p}(\tau + (a - t))}{\pi(\tau + (a - t))} da \quad (t \in [(j + 1)a_b, (j + 2)a_b]).$$

Moreover, by using (3.6) it is not difficult to check that for every $t \in [(j + 1)a_b, (j + 2)a_b]$,

$$(3.17) \quad \int_t^{t+a_+ - \tau} \beta(a)(\Phi_t u)(a) da = \int_t^{t+a_+ - \tau} \beta(a)\pi(a) \frac{\tilde{p}(\tau + (a - t))}{\pi(\tau + (a - t))} da.$$

Using (3.15), (3.16), and (3.17) we have

$$(3.18) \quad \tilde{b}(t) = \int_{a_b}^{t+a_+ - \tau} \beta(a)\pi(a) \frac{\tilde{p}(\tau + (a - t))}{\pi(\tau + (a - t))} da = \chi(t) \quad (t \in [(j + 1)a_b, (j + 2)a_b]).$$

Consequently, if $(j + 1)a_b < t - a \leq (j + 2)a_b$, we obtain from (2.9), (3.13), and (3.18) that

$$\begin{aligned} (\Phi_t u)(a) = \pi(a)\chi(t - a) & \left[1 - \frac{\int_0^a \frac{m(s)}{\pi(s)} ds}{\int_0^{\tau - (t - a)} \frac{m(s)}{\pi(s)} ds} \right] \\ & + \pi(a) \frac{\int_0^a \frac{m(x)}{\pi(x)} dx}{\int_0^{\tau - (t - a)} \frac{m(x)}{\pi(x)} dx} \frac{\tilde{p}(\tau - (t - a))}{\pi(\tau - (t - a))}. \end{aligned}$$

The above formula implies that $(\Phi_\tau u)(a) = \tilde{p}(a)$ for almost every $a \in (\tau - (j + 2)a_b, \tau - (j + 1)a_b)$, which ends the induction proof. \square

Remark 3.3. Given $\tau \in]a_+ - a_0, a_+[$ and $\tilde{p} \in \mathcal{R}$, the proof of Theorem 3.2 gives an explicit formula for a control $u \in L^2([0, \tau]; L^2[0, a_+])$ satisfying $\Phi_\tau u = \tilde{p}$. More precisely, for almost every $(t, a) \in Q_\tau$ we have

$$u(t, a) = \begin{cases} 0 & \text{if } t - a < \tau - a_+, \\ \frac{\tilde{p}(\tau + (a - t))}{\pi(\tau + (a - t))} & \text{if } \tau - a_+ \leq t - a \leq 0, \\ \frac{\tilde{p}(\tau - (t - a)) - \pi(\tau - (t - a))\chi(t - a)}{\pi(\tau - (t - a)) \int_0^{\tau - (t - a)} \frac{m(x)}{\pi(x)} dx} & \text{if } 0 < t - a \leq \tau, \end{cases}$$

where

$$\chi(s) = \int_{a_b}^{s+a_+ - \tau} \beta(x)\pi(x) \frac{\tilde{p}(\tau - (s - x))}{\pi(\tau - (s - x))} dx \quad (s \in [0, \tau]).$$

Moreover, for almost every $(t, a) \in Q_\tau$, we have

$$(\Phi_t u)(a) = \begin{cases} 0 & \text{if } t - a < \tau - a_\dagger, \\ \pi(a) \frac{\int_{a-t}^a \frac{m(s)}{\pi(s)} ds}{\int_{a-t}^{\tau+(a-t)} \frac{m(s)}{\pi(s)} ds} \frac{\tilde{p}(\tau + (a-t))}{\pi(\tau + (a-t))} & \text{if } \tau - a_\dagger \leq t - a \leq 0, \\ \pi(a) \chi(t-a) \left[1 - \frac{\int_0^a \frac{m(s)}{\pi(s)} ds}{\int_0^{\tau-(t-a)} \frac{m(s)}{\pi(s)} ds} \right] \\ + \pi(a) \frac{\int_0^a \frac{m(x)}{\pi(x)} dx}{\int_0^{\tau-(t-a)} \frac{m(x)}{\pi(x)} dx} \frac{\tilde{p}(\tau - (t-a))}{\pi(\tau - (t-a))} & \text{if } 0 < t - a \leq \tau. \end{cases}$$

Remark 3.4. For every $\varphi \in \mathcal{D}(A)$ with $\frac{\varphi}{\pi} \in X$, we have $\varphi \in \mathcal{R}$. Indeed, for every $\varphi \in \mathcal{D}(A)$, there exists $f \in X$ such that $\varphi' + \mu\varphi = f$, so that assuming the condition $\frac{f}{\pi} \in X$, for almost every $a \in (0, a_\dagger)$ we have

$$(3.19) \quad \begin{aligned} \frac{\varphi(a)}{\pi(a)} &= \varphi(0) + \int_0^a \frac{\varphi'(s)\pi(s) + \mu(s)\pi(s)\varphi(s)}{\pi^2(s)} ds \\ &= \int_0^{a_\dagger} \beta(a)\varphi(a) da + \int_0^a \frac{f(s)}{\pi(s)} ds. \end{aligned}$$

Denoting by $I := \int_0^{a_\dagger} \beta(a)\varphi(a) da$ and $i(a) = \int_0^{a_\dagger-a} \beta(x) \frac{\pi(x)}{\pi(a+x)} \varphi(a+x) dx$, it follows from (3.19) that we have

$$\frac{\varphi(a)}{\pi(a)} - \int_0^{a_\dagger-a} \beta(x) \frac{\pi(x)}{\pi(a+x)} \varphi(a+x) dx = \int_0^a \frac{f(s)}{\pi(s)} ds + I - i(a).$$

Let us show that the map

$$(3.20) \quad a \mapsto \frac{\int_0^a \frac{f(s)}{\pi(s)} ds + I - i(a)}{a}$$

is in $L^2[0, a_\dagger]$. By Hardy's inequality (see [12, p. 240]), the map $a \mapsto \frac{\int_0^a \frac{f(s)}{\pi(s)} ds}{a}$ lies in $L^2[0, a_\dagger]$. For the second term in the numerator of (3.20) we note that for almost every $a \in (0, a_\dagger)$ we have

$$\frac{d}{da}(I - i(a)) = -\beta(a_\dagger - a)\pi(a_\dagger - a) \left(\varphi(0) + \int_0^{a_\dagger} \frac{f(s)}{\pi(s)} ds \right) - \int_0^{a_\dagger-a} \beta(x)\pi(x) \frac{f(a+x)}{\pi(a+x)} dx,$$

so that the map $a \mapsto \frac{d}{da}(I - i(a))$ is in $L^2[0, a_\dagger]$. It follows that the map $F : a \mapsto I - i(a)$ satisfies $F \in H^1[0, a_\dagger]$ with $F(0) = 0$, so that from Hardy's inequality (see [12, p. 240]) the map $a \mapsto \frac{F(a)}{a}$ is in $L^2[0, a_\dagger]$. Consequently, from (3.20) we have $\varphi \in \mathcal{R}$.

4. Some controllability results. In this section we investigate some controllability properties of system (1.1).

We first note that the following null controllability result holds.

PROPOSITION 4.1. *For every $\tau > a_{\dagger} - a_0$, we have $\text{Ran } \mathbb{T}_{\tau} \subset \mathcal{R}$, where \mathcal{R} has been defined in (3.2). Consequently, if β satisfies (1.4), then system (1.1) is null controllable in any time $\tau > a_{\dagger} - a_0$.*

Proof. Without loss of generality, we can assume that $a_{\dagger} - a_0 < \tau < a_{\dagger}$. Indeed, for $\tau \geq a_{\dagger}$, the result follows using the semigroup property and replacing p_0 by $\mathbb{T}_{\varepsilon} p_0$ for some convenient $\varepsilon > 0$. Let $\tilde{p} \in \text{Ran } \mathbb{T}_{\tau}$. Then from (2.4) there exists $p_0 \in X$ such that

$$(4.1) \quad \tilde{p}(a) = \begin{cases} \frac{\pi(a)}{\pi(a-\tau)} p_0(a-\tau) & \text{if } \tau \leq a, \\ \pi(a)b(\tau-a) & \text{if } \tau > a \end{cases} \quad (a \in [0, a_{\dagger}] \text{ a.e.}),$$

with $b(t) = \int_0^{a_{\dagger}} \beta(a)(\mathbb{T}_t p_0)(a) da$. Consequently, for almost every $a \in (0, \tau)$ we have $\frac{\tilde{p}(a)}{\pi(a)} = b(\tau - a)$, and for almost every $a \in (\tau, a_{\dagger})$ we have $\frac{\tilde{p}(a)}{\pi(a)} = \frac{p_0(a-\tau)}{\pi(a-\tau)}$. Hence, the function $a \mapsto \frac{\tilde{p}}{\pi}$ is in $L^2[0, a_{\dagger}]$. From this it is easy to check that

$$(4.2) \quad a \mapsto \frac{\frac{\tilde{p}(a)}{\pi(a)} - \int_0^{a_{\dagger}-a} \beta(x) \frac{\pi(x)}{\pi(a+x)} \tilde{p}(a+x) dx}{\int_0^a \frac{m(x)}{\pi(x)} dx}$$

is in $L^2[\varepsilon, a_{\dagger}]$ for every $\varepsilon > 0$. To show that $\tilde{p} \in \mathcal{R}$, we still have to check that the map defined in (4.2) is in $L^2[0, \varepsilon]$ for every $\varepsilon > 0$ small enough. To this aim, applying first (2.4) we note that, for almost every $a \in (0, \varepsilon)$,

$$\begin{aligned} & \frac{\tilde{p}(a)}{\pi(a)} - \int_0^{a_{\dagger}-a} \beta(x) \frac{\pi(x)}{\pi(a+x)} \tilde{p}(a+x) dx \\ &= b(\tau - a) - \int_0^{\tau-a} \beta(x)\pi(x)b(\tau - (a+x)) dx - \int_{\tau-a}^{a_{\dagger}-a} \beta(x)\pi(x) \frac{p_0(a+x-\tau)}{\pi(a+x-\tau)} dx. \end{aligned}$$

Moreover, using (2.5) it follows that for every $a \in (0, \varepsilon)$ we have

$$b(\tau - a) = \int_0^{\tau-a} \beta(x)\pi(x)b(\tau - (a+x)) dx + \int_{\tau-a}^{a_{\dagger}} \beta(x)\pi(x) \frac{p_0(a+x-\tau)}{\pi(a+x-\tau)} dx.$$

We can thus combine the last two formulas to obtain that for almost every $a \in (0, \varepsilon)$ we have

$$\frac{\tilde{p}(a)}{\pi(a)} - \int_0^{a_{\dagger}-a} \beta(x) \frac{\pi(x)}{\pi(a+x)} \tilde{p}(a+x) dx = \int_{a_{\dagger}-a}^{a_{\dagger}} \beta(x)\pi(x) \frac{p_0(a+x-\tau)}{\pi(a+x-\tau)} dx.$$

Since $\beta \in L^{\infty}[0, a_{\dagger}]$ and π is a decreasing function, it follows that for almost every $a \in (0, \varepsilon)$,

$$(4.3) \quad \left| \frac{\tilde{p}(a)}{\pi(a)} - \int_0^{a_{\dagger}-a} \beta(x) \frac{\pi(x)}{\pi(a+x)} \tilde{p}(a+x) dx \right| \leq \|\beta\|_{L^{\infty}[0, a_{\dagger}]} \int_{a_{\dagger}-\tau}^{a_{\dagger}-\tau+a} |p_0(s)| ds.$$

On the other hand, Hardy’s inequality (see, for instance, [12, p. 240]) ensures that

$$a \mapsto \frac{\int_0^a |p_0(s + (a_{\dagger} - \tau))| ds}{a} \in L^2[0, \varepsilon],$$

and hence $\tilde{p} \in \mathcal{R}$.

To obtain the null controllability assertion it suffices to combine the inclusions $\text{Ran } \mathbb{T}_\tau \subset \mathcal{R}$, which we have just proved, and $\mathcal{R} \subset \text{Ran } \Phi_\tau$, which was proved in Theorem 3.2. \square

The result below and the following one answer an open question raised in [8], and it essentially asserts that, with an extra assumption, the system can be “positively” steered from any positive initial state in X to any positive state in \mathcal{R} by means of an L^2 control. More precisely, we have the following.

THEOREM 4.2. *Let the assumptions of Theorem 1.1 be satisfied. Let $p_0 \in X$ and $\tilde{p} \in \mathcal{R}$. Then for every $\tau > a_+ - a_0$ there exists a control $u \in L^2([0, \tau]; L^2[0, a_+])$ such that the controlled solution p of (1.1) satisfies*

$$p(\tau, a) = \tilde{p}(a) \quad (a \in (0, a_+) \text{ a.e.}).$$

Moreover, if $p_0(a) \geq 0$ and $\tilde{p}(a) \geq 0$ for almost every $a \in (0, a_+)$, then the control can be chosen such that the controlled state trajectory remains positive, i.e., such that for every $t \in [0, \tau]$ we have

$$p(t, a) \geq 0 \quad (a \in [0, a_+] \text{ a.e.}).$$

Proof. We give the proof in the case $\tau \in (a_+ - a_0, a_+)$.

We set $z = p - \tilde{p}$. The aim is to find a control function $u \in L^2([0, \tau]; L^2[0, a_+])$ such that $z(\tau) = 0$ and $p(t, a) \geq 0$ for almost every $(t, a) \in Q_\tau$. To achieve this goal we set

$$z(t) = \mathbb{T}_t p_0 + \Phi_t u - \tilde{p} \quad (t \geq 0).$$

Since from Theorem 3.2 and Proposition 4.1 we have $\tilde{p} - \mathbb{T}_\tau p_0 \in \mathcal{R}$ we can apply the construction in Theorem 3.2 to obtain a control $u \in L^2([0, \tau]; L^2[0, a_+])$ such that

$$\Phi_\tau u = \tilde{p} - \mathbb{T}_\tau p_0.$$

The corresponding state trajectory will satisfy, in particular, the condition $p(\tau, \cdot) = \tilde{p}$.

To prove the positivity of the state trajectory constructed above we first assume that p_0 and \tilde{p} are continuous on $[0, a_+]$. Then the control u described in Remark 3.3 is of the form $u(t, a) = v(t - a)$, with v continuous on each one of the intervals $[-a_+, \tau - a_+]$, $[\tau - a_+, 0]$, and $[0, \tau]$. Moreover, the corresponding state trajectory p , also described in Remark 3.3, is continuous on each one of the domains Q_τ^1 , Q_τ^2 , and Q_τ^3 , where

$$\begin{aligned} Q_\tau^1 &= \{(t, a) \in Q_\tau \mid -a_+ \leq t - a < \tau - a_+\}, \\ Q_\tau^2 &= \{(t, a) \in Q_\tau \mid \tau - a_+ \leq t - a \leq 0\}, \\ Q_\tau^3 &= \{(t, a) \in Q_\tau \mid 0 < t - a \leq \tau\}. \end{aligned}$$

Since the initial condition p_0 is assumed positive and the control u vanishes on Q_τ^1 , the solution p of the controlled system (1.1) is positive on Q_τ^1 . Indeed, from (2.9) we have

$$(4.4) \quad p(t, a) = \frac{\pi(a)}{\pi(a-t)} p_0(a-t) \quad ((t, a) \in Q_\tau^1).$$

Moreover, if $(t, a) \in Q_\tau^2$ with $a \geq a_0$, using (2.9) we have

$$(4.5) \quad p(t, a) = \frac{\pi(a)}{\pi(a-t)} p_0(a-t) + v(t-a) \int_{a-t}^{a_0} \frac{\pi(a)}{\pi(s)} ds \quad ((t, a) \in Q_\tau^2, a \geq a_0).$$

It follows from the above formula that for every $(t, a) \in Q_\tau^2$ with $a \geq a_0$ we have

$$(4.6) \quad \tilde{p}(\tau - (t - a)) = p(\tau, \tau - (t - a)) = \frac{\pi(\tau - (t - a))}{\pi(a - t)} p_0(a - t) + v(t - a) \int_{a-t}^{a_0} \frac{\pi(\tau - (t - a))}{\pi(s)} ds \quad ((t, a) \in Q_\tau^2, a \geq a_0).$$

By combining (4.5) and (4.6) we get

$$(4.7) \quad p(t, a) = \frac{\pi(a)}{\pi(\tau - (t - a))} \tilde{p}(\tau - (t - a)) \quad ((t, a) \in Q_\tau^2, a \geq a_0).$$

Since the target \tilde{p} is assumed positive, it follows from (4.7) that the solution p of the controlled system (1.1) is positive on the subdomain $\{(t, a) \in Q_\tau^2, a \geq a_0\}$.

If $(t, a) \in Q_\tau^3$ with $a \geq a_0$, using (2.9) we have

$$(4.8) \quad p(t, a) = \pi(a) \tilde{b}(t - a) + v(t - a) \int_0^{a_0} \frac{\pi(a)}{\pi(s)} ds \quad ((t, a) \in Q_\tau^3, a \geq a_0).$$

It follows from the above formula that for every $(t, a) \in Q_\tau^3$ with $a \geq a_0$ we have

$$(4.9) \quad \tilde{p}(\tau - (t - a)) = \pi(\tau - (t - a)) \tilde{b}(t - a) + v(t - a) \int_0^{a_0} \frac{\pi(\tau - (t - a))}{\pi(s)} ds \quad ((t, a) \in Q_\tau^3, a \geq a_0),$$

so that using (4.8) and (4.9) we get

$$(4.10) \quad p(t, a) = \frac{\pi(a)}{\pi(\tau - (t - a))} \tilde{p}(\tau - (t - a)) \quad ((t, a) \in Q_\tau^3, a \geq a_0).$$

Since the target \tilde{p} is assumed positive, it follows from (4.10) that the solution p of the controlled system (1.1) is positive on the subdomain $\{(t, a) \in Q_\tau^3, a \geq a_0\}$. In particular, using (4.7) and (4.10), the solution p is positive along the red curve represented in Figure 1.

It remains to show the positivity of the controlled solution p into the subregions $\{(t, a) \in Q_\tau^2, a \leq a_0\}$ and $\{(t, a) \in Q_\tau^3, a \leq a_0\}$. To this aim, note first that if $(t, a) \in Q_\tau^2$ with $a \leq a_0$, then from (2.9) we have

$$(4.11) \quad p(t, a) = \pi(a) \left(\frac{p_0(a - t)}{\pi(a - t)} + v(t - a) \int_{a-t}^a \frac{m(s)}{\pi(s)} ds \right) \quad ((t, a) \in Q_\tau^2, a \leq a_0).$$

Let us show that the density $p(t, a)$ is positive along each characteristic line $t - a = c$, with $c \in [\tau - a_+, 0]$ and $a \leq a_0$. Note first that if $c \in [\tau - a_+, 0]$, then for every $(t, a) \in Q_\tau^2$ with $a \leq a_0$ and $t - a = c$, from (4.11) we have

$$(4.12) \quad p(a + c, a) = \pi(a) \left(\frac{p_0(-c)}{\pi(-c)} + v(c) \int_{-c}^a \frac{m(s)}{\pi(s)} ds \right).$$

If $v(c) \geq 0$, then from the above formula (4.12) the density $p(t, a)$ is clearly positive along the characteristic line $t - a = c$. If $v(c) < 0$, the control acts like a retrieval

of individuals so that along the characteristic line $t - a = c$ the population density decreases and must achieve its minimum value on the red vertical line represented in Figure 1, the minimum of which is positive due to (4.7) and the fact that the solution p is continuous on Q_τ^2 . In fact, with c fixed in $[\tau - a_\dagger, 0]$, in the case where $v(c) < 0$, the quantity

$$\frac{p_0(-c)}{\pi(-c)} + v(c) \int_{-c}^a \frac{m(s)}{\pi(s)} ds$$

is a decreasing expression of a which attains its positive minimum for $a = a_0$. It follows that $p(t, a) \geq 0$ for almost every $(t, a) \in Q_\tau^2$ with $a \leq a_0$.

Similar arguments can be used to show the positivity of the solution p on $\{(t, a) \in Q_\tau^3, a \leq a_0\}$. Indeed, if $(t, a) \in Q_\tau^3$ with $a \leq a_0$, then from (2.9) we have

$$(4.13) \quad p(t, a) = \pi(a) \left(\tilde{b}(t - a) + v(t - a) \int_0^a \frac{m(s)}{\pi(s)} ds \right) \quad ((t, a) \in Q_\tau^3, a \leq a_0),$$

with

$$(4.14) \quad \tilde{b}(t) = \int_{a_b}^{a_\dagger} \beta(a) p(t, a) da \quad (t \in [0, \tau]).$$

From (4.4), (4.7), (4.10), and (4.14) it follows that we have $\tilde{b}(t) \geq 0$ for every $t \in [0, \tau]$, since the density $p(t, a)$ is positive for $a \geq a_b \geq a_0$. Distinguish again the cases $v(t - a) \geq 0$ and $v(t - a) < 0$ in (4.13), and using the positivity and the continuity of p along the red and green lines represented in Figure 1 provides the positivity of the controlled solution p on the subdomain $\{(t, a) \in Q_\tau^3, a \leq a_0\}$.

The case where $p_0 \in X$ and $\tilde{p} \in \mathcal{R}$ are not necessarily continuous can be deduced from a density argument. Indeed, it suffices to note that it follows from Remark 3.3 that if $(p_{0,n})_{n \geq 1} \subset X$ and $(\tilde{p}_n)_{n \geq 1} \subset \mathcal{R}$ are such that $p_{0,n} \rightarrow p_0$ and $\tilde{p}_n \rightarrow \tilde{p}$, then the corresponding state trajectories satisfy $p_n \rightarrow p$ in $L^2(Q_\tau)$. \square

Remark 4.3. The control and the state trajectory constructed in the above proof can be explicitly written in terms of p_0 and \tilde{p} . Indeed, applying the formulas in Remark 3.3, with $\tilde{p} - \mathbb{T}_\tau p_0$ instead of \tilde{p} and after some calculations, it can be checked that for almost every $(t, a) \in Q_\tau$ we have

$$u(t, a) = \begin{cases} 0 & \text{if } t - a < \tau - a_\dagger, \\ \frac{\tilde{p}(\tau + (a - t)) - p_0(a - t)}{\pi(\tau + (a - t)) - \pi(a - t)} & \text{if } \tau - a_\dagger \leq t - a \leq 0, \\ \frac{\int_{a-t}^{\tau+(a-t)} \frac{m(x)}{\pi(x)} dx}{\tilde{p}(\tau - (t - a)) - \pi(\tau - (t - a))\eta(t - a)} & \text{if } 0 < t - a \leq \tau, \\ \frac{\tilde{p}(\tau - (t - a)) - \pi(\tau - (t - a))\eta(t - a)}{\pi(\tau - (t - a)) \int_0^{\tau-(t-a)} \frac{m(x)}{\pi(x)} dx} & \end{cases}$$

where

$$\eta(s) = \int_{a_b}^{s+a_\dagger-\tau} \beta(x) \pi(x) \frac{\tilde{p}(\tau - (s - x))}{\pi(\tau - (s - x))} dx + \int_{s+a_\dagger-\tau}^{a_\dagger} \beta(x) \pi(x) \frac{p_0(x - s)}{\pi(x - s)} dx \quad (s \in [0, \tau]).$$

Moreover, for almost every $(t, a) \in Q_\tau$, we have

$$p(t, a) = \begin{cases} \frac{\pi(a)}{\pi(a-t)} p_0(a-t) & \text{if } t-a < \tau - a_+, \\ \frac{\pi(a)}{\pi(a-t)} p_0(a-t) \left[1 - \frac{\int_{a-t}^a \frac{m(x)}{\pi(x)} dx}{\int_{a-t}^{a+(\tau-t)} \frac{m(x)}{\pi(x)} dx} \right] \\ + \pi(a) \frac{\int_{a-t}^a \frac{m(s)}{\pi(s)} ds}{\int_{a-t}^{\tau+(a-t)} \frac{m(s)}{\pi(s)} ds} \frac{\tilde{p}(\tau + (a-t))}{\pi(\tau + (a-t))} & \text{if } \tau - a_+ \leq t-a \leq 0, \\ \pi(a) \eta(t-a) \left[1 - \frac{\int_0^a \frac{m(s)}{\pi(s)} ds}{\int_0^{\tau-(t-a)} \frac{m(s)}{\pi(s)} ds} \right] \\ + \pi(a) \frac{\int_0^a \frac{m(x)}{\pi(x)} dx}{\int_0^{\tau-(t-a)} \frac{m(x)}{\pi(x)} dx} \frac{\tilde{p}(\tau - (t-a))}{\pi(\tau - (t-a))} & \text{if } 0 < t-a \leq \tau. \end{cases}$$

We show below that the result in Theorem 4.2 applies for target states that are stationary solutions of (1.1), which is the case investigated in [8].

Proof of Theorem 1.1. According to Theorem 4.2, we only have to show that any solution \tilde{p} of (1.2) satisfies $\tilde{p} \in \mathcal{R}$.

From [8] it follows that any solution \tilde{p} of (1.2) is given by

$$(4.15) \quad \tilde{p}(a) = C\pi(a) + \int_0^a \frac{\pi(a)}{\pi(\sigma)} m(\sigma)w(\sigma) d\sigma \quad (a \in (0, a_+)),$$

where C is determined through

$$(4.16) \quad (1 - R)C = \int_0^{a_+} \beta(x)\pi(x) \int_0^x \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma dx,$$

and R is the reproductive number of the population given by

$$R = \int_0^{a_+} \beta(a)\pi(a) da.$$

From (4.15) it follows that for almost every $a \in (0, a_+)$ we have

$$\frac{\tilde{p}(a)}{\pi(a)} = C + \int_0^a \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma,$$

so that $\frac{\tilde{p}}{\pi} \in L^2[0, a_+]$. Moreover, for almost every $a \in (0, a_+)$ we have from (4.15) that

$$\begin{aligned} \frac{\tilde{p}(a)}{\pi(a)} - \int_0^{a_+-a} \beta(x) \frac{\pi(x)}{\pi(a+x)} \tilde{p}(a+x) dx \\ = C + \int_0^a \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma - \int_0^{a_+-a} \beta(x)\pi(x) \left[C + \int_0^{a+x} \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma \right] dx, \end{aligned}$$

so that for almost every $a \in (0, a_+)$ we have

$$(4.17) \quad \begin{aligned} \frac{\tilde{p}(a)}{\pi(a)} - \int_0^{a_+ - a} \beta(x) \frac{\pi(x)}{\pi(a+x)} \tilde{p}(a+x) dx \\ = \int_0^a \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma + C \left(1 - \int_0^{a_+ - a} \beta(x) \pi(x) dx \right) \\ - \int_0^{a_+ - a} \beta(x) \pi(x) \int_0^{a+x} \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma dx. \end{aligned}$$

Define $\mathcal{G} := \int_0^{a_+} \beta(x) \pi(x) \int_0^x \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma dx$ such that from (4.16) we have $(1-R)C = \mathcal{G}$. Define

$$r(a) := \int_0^{a_+ - a} \beta(x) \pi(x) dx \quad \text{and} \quad g(a) := \int_0^{a_+ - a} \beta(x) \pi(x) \int_0^{a+x} \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma dx.$$

Then (4.17) yields that for almost every $a \in (0, a_+)$ we have

$$(4.18) \quad \begin{aligned} \frac{\tilde{p}(a)}{\pi(a)} - \int_0^{a_+ - a} \beta(x) \frac{\pi(x)}{\pi(a+x)} \tilde{p}(a+x) dx \\ = \int_0^a \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma + C(1 - r(a)) - g(a) \\ = \int_0^a \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma + C(R - r(a)) + (\mathcal{G} - g(a)). \end{aligned}$$

Let us show that the map

$$(4.19) \quad a \mapsto \frac{\int_0^a \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma + C(R - r(a)) + (\mathcal{G} - g(a))}{a}$$

is in $L^2[0, a_+]$. By Hardy's inequality (see [12, p. 240]), the map

$$a \mapsto \frac{\int_0^a \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma}{a}$$

lies in $L^2[0, a_+]$. For the second term at the numerator in (4.19) we note that for almost every $a \in (0, a_+)$ we have

$$\begin{aligned} |R - r(a)| &= \int_{a_+ - a}^{a_+} \beta(x) \pi(x) dx \\ &\leq a \|\beta\|_{L^\infty[0, a_+]}, \end{aligned}$$

so that the map $a \mapsto \frac{R - r(a)}{a}$ is in $L^2[0, a_+]$.

We still have to tackle the third term at the numerator in (4.19). Toward this goal we note that for almost every $a \in (0, a_+)$ we have

$$(4.20) \quad \begin{aligned} \frac{d}{da} (\mathcal{G} - g(a)) &= -\beta(a_+ - a) \pi(a_+ - a) \int_0^{a_+} \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma \\ &\quad - \int_0^{a_+ - a} \beta(x) \pi(x) \frac{m(a+x)}{\pi(a+x)} w(a+x) dx, \end{aligned}$$

so that the map $a \mapsto \frac{d}{da}(\mathcal{G} - g(a))$ is in $L^2[0, a_+]$. It follows that the map $F : a \mapsto \mathcal{G} - g(a)$ satisfies $F \in \dot{H}^1[0, a_+]$ with $F(0) = 0$, so that from Hardy’s inequality (see [12, p. 240]) the map $a \mapsto \frac{F(a)}{a}$ is in $L^2[0, a_+]$. Consequently, from (4.19) we have $\tilde{p} \in \mathcal{R}$. \square

Remark 4.4. The method used above to prove Theorem 1.1 cannot be adapted to the case in which the state space $X = L^2[0, a_+]$ is substituted by $X = L^1[0, a_+]$. Indeed, if w and \tilde{p} are in L^1 , the fact that \tilde{p} is in \mathcal{R} no longer holds in general. Indeed, the map

$$a \mapsto \frac{\int_0^a \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) \, d\sigma}{a}$$

does not necessarily belong to $L^1[0, a_+]$ for any $w \in L^1[0, a_+]$ (see Hardy’s inequality [12]). However, the inclusion $\text{Ran } \mathbb{T}_\tau \subset \text{Ran } \Phi_\tau$ remains true for every $\tau > a_+ - a_0$; see Appendix A.

5. A singular perturbation problem. In this section we study the behavior of the null controls constructed in section 4, together with the behavior of the corresponding state trajectories, when the age interval in which the control is active shrinks to the singleton $\{0\}$. Not surprisingly, we show that the null controls converge to a null control for a system in which the input appears in an additive manner in the formula giving the birth rate. The control in this limit system can be seen as an “impulse” (in the age domain) type of control or, simpler, as a direct control of the birth rate. Similar singular perturbation problems for the wave or for the Schrödinger equations have been studied in Fabre [9], Fabre and Puel [10, 11], and Joly [15].

To be more precise, we consider the control system (1.1) for $a_0 = \varepsilon \in (0, a_+)$. According to Proposition 4.1, given $\varepsilon > 0$, for every $p_0 \in X = L^2[0, a_+]$ there exists $u^\varepsilon \in L^2([0, a_+]; X)$ such that the solution of

$$(5.1) \quad \begin{cases} \dot{p}^\varepsilon(t, a) + \frac{\partial p^\varepsilon}{\partial a}(t, a) + \mu(a)p^\varepsilon(t, a) = \mathbb{1}_{[0, \varepsilon]}(a)u^\varepsilon(t, a), & (t, a) \in (0, a_+)^2, \\ p^\varepsilon(t, 0) = \int_0^{a_+} \beta(a)p^\varepsilon(t, a) \, da, & t \in (0, a_+), \\ p^\varepsilon(0, a) = p_0(a), & a \in (0, a_+), \end{cases}$$

satisfies

$$(5.2) \quad p^\varepsilon(a_+, a) = 0 \quad (a \in (0, a_+)).$$

Moreover, by a slight variation of the calculations used to derive the formula in Remark 4.3, it is easy to check that the control u^ε defined by

$$(5.3) \quad u^\varepsilon(t, a) = \begin{cases} - \frac{\int_{t-a}^{a_+} \beta(x)\pi(x) \frac{p_0(x-(t-a))}{\pi(x-(t-a))} \, dx}{\int_0^{a_+-(t-a)} \frac{\mathbb{1}_{[0, \varepsilon]}(x)}{\pi(x)} \, dx} & ((t, a) \in (0, a_+)^2, t - a \geq 0), \\ 0 & ((t, a) \in (0, a_+)^2, t - a < 0) \end{cases}$$

satisfies the above conditions. With u_ε defined by (5.3), the controlled density p^ε

solution of (5.1) is given by

$$(5.4) \quad p^\varepsilon(t, a) = \begin{cases} \frac{\pi(a)}{\pi(a-t)} p_0(a-t) & \text{if } t-a < 0, \\ \pi(a) \eta(t-a) \left[1 - \frac{\int_0^a \frac{\mathbb{1}_{[0,\varepsilon]}(x)}{\pi(x)} dx}{\int_0^{a_+-(t-a)} \frac{\mathbb{1}_{[0,\varepsilon]}(x)}{\pi(x)} dx} \right] & \text{if } 0 < t-a \leq a_+, \end{cases}$$

where

$$\eta(s) = \int_s^{a_+} \beta(x) \pi(x) \frac{p_0(x-s)}{\pi(x-s)} dx \quad (s \in [0, a_+]).$$

We also consider the following boundary controlled system, already studied in [8]:

$$(5.5) \quad \begin{cases} \dot{p}(t, a) + \frac{\partial p}{\partial a}(t, a) + \mu(a)p(t, a) = 0, & (t, a) \in (0, a_+)^2, \\ p(t, 0) = \int_0^{a_+} \beta(a)p(t, a) da + v(t), & t \in (0, a_+), \\ p(0, a) = p_0(a), & a \in (0, a_+), \end{cases}$$

where $v \in L^2([0, a_+])$. Using a recursive argument as in the proof of Theorem 3.2, it can be easily shown that, taking

$$(5.6) \quad v(t) := - \int_t^{a_+} \beta(a) \frac{\pi(a)}{\pi(a-t)} p_0(a-t) da \quad (t \in [0, a_+]),$$

the solution p of (5.5) satisfies

$$(5.7) \quad p(a_+, a) = 0 \quad (a \in (0, a_+)).$$

More precisely, with v defined by (5.6), the controlled density p solution of (5.5) is given by

$$(5.8) \quad p(t, a) = \begin{cases} \frac{\pi(a)}{\pi(a-t)} p_0(a-t) & \text{if } t \leq a, \\ 0 & \text{if } t > a. \end{cases}$$

The main result of this section is as follows.

THEOREM 5.1. *Given $p_0 \in L^2([0, a_+])$, consider the family of distributed controls $(u^\varepsilon)_{\varepsilon \in (0, a_+)}$ in $L^2([0, a_+]; L^2([0, a_+]))$ and the boundary control v in $L^2[0, a_+]$, respectively, defined by (5.3) and (5.6). Then we have*

1. $\lim_{\varepsilon \rightarrow 0^+} u^\varepsilon \mathbb{1}_{[0,\varepsilon]} = v \delta_0$ weakly in $L^2([0, a_+], \mathcal{D}(A^*)')$, where δ_0 stands for the Dirac mass concentrated at the origin;
2. $\lim_{\varepsilon \rightarrow 0^+} \|p^\varepsilon - p\|_{C([0, a_+]; L^2([0, a_+]))} = 0$, where p is the solution of (5.5).

Proof. Let $\varepsilon \in (0, a_+/2)$. It is easy to check that every function in $\mathcal{D}(A^*)$ belongs to $H^1(0, a^*)$ for every $a^* \in (0, a_+)$. In particular, for every $\varphi \in L^2([0, a_+], \mathcal{D}(A^*))$ and every $t \in [0, a_+]$, we have $\varphi(t, \cdot) \in L^\infty(0, a_+/2)$.

We introduce the following notation:

$$(5.9) \quad f(t, a) := \begin{cases} - \int_{t-a}^{a_+} \beta(x) \pi(x) \frac{p_0(x-(t-a))}{\pi(x-(t-a))} dx & ((t, a) \in (0, a_+)^2, t-a \geq 0), \\ 0 & ((t, a) \in (0, a_+)^2, t-a < 0), \end{cases}$$

and

$$(5.10) \quad k^\varepsilon(t, a) := \int_0^{a_+ - (t-a)} \frac{\mathbb{1}_{[0, \varepsilon]}(x)}{\pi(x)} dx \quad ((t, a) \in (0, a_+)^2),$$

so that from (5.3), (5.9), and (5.10) we have

$$u^\varepsilon(t, a) = \frac{f(t, a)}{k^\varepsilon(t, a)} \quad ((t, a) \in (0, a_+)^2).$$

Let $\varphi \in L^2([0, a_+], \mathcal{D}(A^*))$, and denote

$$(5.11) \quad g_\varepsilon(t) := \int_0^{a_+} \mathbb{1}_{[0, \varepsilon]}(a) u^\varepsilon(t, a) \varphi(t, a) da = \int_0^\varepsilon \frac{f(t, a)}{k^\varepsilon(t, a)} \varphi(t, a) da \quad (t \in [0, a_+]).$$

The aim is to show that $\lim_{\varepsilon \rightarrow 0} g_\varepsilon = v\varphi(\cdot, 0)$ in $L^2[0, a_+]$. To this aim, we use Lebesgue’s dominated convergence theorem. First, let us show that for every $t \in [0, a_+]$ we have the following pointwise convergence:

$$(5.12) \quad \lim_{\varepsilon \rightarrow 0^+} g_\varepsilon(t) = v(t)\varphi(t, 0) \quad (t \in [0, a_+]).$$

For every $t \in [0, a_+]$, take $\varepsilon < a_+ - t$ small enough so that $\varepsilon < a_+ - (t - a)$ for every $a \in (0, \varepsilon)$. It follows from (5.10) that we have

$$(5.13) \quad k^\varepsilon(t, a) = \int_0^\varepsilon \frac{dx}{\pi(x)} \quad (a \in (0, \varepsilon)),$$

so that from (5.11) and (5.13) we have

$$g_\varepsilon(t) = \frac{1}{\int_0^\varepsilon \frac{1}{\pi(x)} dx} \int_0^\varepsilon f(t, a) \varphi(t, a) da,$$

and, using the last formula and the fact that $f(t, 0) = v(t)$, we have

$$(5.14) \quad \lim_{\varepsilon \rightarrow 0} g_\varepsilon(t) = v(t)\varphi(t, 0) \quad (t \in [0, a_+]).$$

For $t = a_+$, it follows from (5.10) that we have

$$(5.15) \quad k^\varepsilon(a_+, a) = \int_0^a \frac{dx}{\pi(x)} \quad (a \in (0, \varepsilon)),$$

so that from (5.11) and (5.15) we have

$$g_\varepsilon(a_+) = \int_0^\varepsilon \frac{f(a_+, a)}{\int_0^a \frac{1}{\pi(x)} dx} \varphi(a_+, a) da,$$

and, using the above formula, together with the fact that $v(a_+) = 0$, we obtain that

$$(5.16) \quad \lim_{\varepsilon \rightarrow 0} g_\varepsilon(a_+) = v(a_+)\varphi(a_+, 0).$$

Consequently, it follows from (5.14) and (5.16) that the pointwise convergence (5.12) holds for every $t \in [0, a_+]$.

Second, let us find a function $h \in L^2[0, a_+]$ such that

$$(5.17) \quad |g_\varepsilon(t)| \leq h(t) \quad (\varepsilon \in (0, a_+/2), t \in [0, a_+]).$$

It is easy to check that for every $t \in [0, a_+ - \varepsilon]$ we have

$$(5.18) \quad k^\varepsilon(t, a) = \int_0^\varepsilon \frac{dx}{\pi(x)} \quad \text{if } a \in (0, \varepsilon),$$

and for every $t \in [a_+ - \varepsilon, a_+]$ we have

$$(5.19) \quad k^\varepsilon(t, a) = \begin{cases} \int_0^{a_+ - (t-a)} \frac{dx}{\pi(x)} & \text{if } a \in (0, \varepsilon + t - a_+), \\ \int_0^\varepsilon \frac{dx}{\pi(x)} & \text{if } a \in (\varepsilon + t - a_+, \varepsilon). \end{cases}$$

It follows from (5.11), (5.18), and (5.19) that we have

$$(5.20) \quad g_\varepsilon(t) = \begin{cases} \frac{1}{\int_0^\varepsilon \frac{1}{\pi(x)} dx} \int_0^\varepsilon f(t, a) \varphi(t, a) da & (t \in [0, a_+ - \varepsilon]), \\ \int_0^{\varepsilon + t - a_+} \frac{f(t, a) \varphi(t, a)}{\int_0^{a_+ - (t-a)} \frac{1}{\pi(x)} dx} da \\ \quad + \frac{1}{\int_0^\varepsilon \frac{1}{\pi(x)} dx} \int_{\varepsilon + t - a_+}^\varepsilon f(t, a) \varphi(t, a) da & (t \in [a_+ - \varepsilon, a_+]). \end{cases}$$

For every $t \in [0, a_+ - \varepsilon]$, since $f \in L^\infty([0, a_+]^2)$ and $\varphi(t, \cdot) \in L^\infty([0, a_+/2])$, from (5.20) we get

$$(5.21) \quad |g_\varepsilon(t)| \leq \|f\|_{L^\infty([0, a_+]^2)} \|\varphi(t, \cdot)\|_{L^\infty([0, a_+/2])}.$$

For every $t \in [a_+ - \varepsilon, a_+]$, from (5.20) we have

$$(5.22) \quad |g_\varepsilon(t)| \leq \int_0^\varepsilon \frac{f(t, a) \varphi(t, a)}{a_+ - (t-a)} da + \frac{1}{\varepsilon} \int_0^\varepsilon f(t, a) \varphi(t, a) da.$$

Since $p_0 \in L^2[0, a_+]$, the Cauchy–Schwarz inequality gives

$$(5.23) \quad |f(t, a)| \leq K \|p_0\|_{L^2(0, a_+)} \sqrt{a_+ - (t-a)}$$

for some constant $K \geq 0$. It follows from (5.22) and (5.23) that we have

$$(5.24) \quad |g_\varepsilon(t)| \leq K \|p_0\|_{L^2(0, a_+)} \|\varphi(t, \cdot)\|_{L^\infty([0, a_+/2])} \int_0^{a_+/2} \frac{da}{\sqrt{a_+ - (t-a)}} \\ + \|f\|_{L^\infty([0, a_+]^2)} \|\varphi(t, \cdot)\|_{L^\infty([0, a_+/2])},$$

and the following majoration holds:

$$(5.25) \quad |g_\varepsilon(t)| \leq 2K \|p_0\|_{L^2(0, a_+)} \|\varphi(t, \cdot)\|_{L^\infty([0, a_+/2])} \sqrt{3a_+/2} + \|f\|_{L^\infty([0, a_+]^2)} \|\varphi(t, \cdot)\|_{L^\infty([0, a_+/2])}.$$

Define, for every $t \in [0, a_+]$,

$$(5.26) \quad h(t) := 2K \|p_0\|_{L^2(0, a_+)} \|\varphi(t, \cdot)\|_{L^\infty([0, a_+/2])} \sqrt{3a_+/2} + \|f\|_{L^\infty([0, a_+])} \|\varphi(t, \cdot)\|_{L^\infty([0, a_+/2])}.$$

It follows from (5.21) and (5.25) that for every $\varepsilon \in (0, a_+/2)$ and every $t \in [0, a_+]$, we have

$$(5.27) \quad |g_\varepsilon(t)| \leq h(t) \quad (\varepsilon \in (0, a_+/2), t \in [0, a_+]),$$

with $h \in L^2([0, a_+])$. The pointwise convergence (5.12) and the above domination (5.27) gives

$$(5.28) \quad \lim_{\varepsilon \rightarrow 0^+} g_\varepsilon = v\varphi(\cdot, 0) \text{ in } L^2[0, a_+],$$

so that we have

$$(5.29) \quad \lim_{\varepsilon \rightarrow 0^+} u^\varepsilon \mathbb{1}_{[0, \varepsilon]} = v\delta_0 \text{ weakly in } L^2([0, a_+], \mathcal{D}(A^*)').$$

Concerning the convergence of the state trajectories, it follows from (5.4) and (5.8) that for every $t \in [0, a_+]$,

$$(5.30) \quad \int_0^{a_+} |p^\varepsilon(t, a) - p(t, a)|^2 da \leq \|\eta\|_{L^\infty[0, a_+]}^2 \int_0^\varepsilon \left(1 - \frac{\int_0^a \frac{\mathbb{1}_{[0, \varepsilon]}(x)}{\pi(x)} dx}{\int_0^{a_+ - (t-a)} \frac{\mathbb{1}_{[0, \varepsilon]}(x)}{\pi(x)} dx} \right)^2 da \leq \|\eta\|_{L^\infty[0, a_+]}^2 \varepsilon.$$

The convergence stated in Theorem 5.1 easily follows from (5.30). □

6. Control of an infection-age model. The methodology developed in the previous sections is essentially applicable to linear systems. However, the fact that controls are explicitly determined by calculations along characteristics allows us, in some particular situations, to tackle nonlinear systems. The aim of this section is thus to provide an example of application of our methods for a nonlinear controlled infection-age model, written as

$$(6.1) \quad \begin{cases} \frac{dS}{dt}(t) = -S(t) \int_0^{a_+} \beta(a)i(t, a) da, \\ \frac{\partial i}{\partial t}(t, a) + \frac{\partial i}{\partial a}(t, a) = -\mu(a)i(t, a) - m(a)u(t, a), \\ i(t, 0) = S(t) \int_0^{a_+} \beta(a)i(t, a) da, \end{cases}$$

with the initial condition

$$S(0) = S_0 \geq 0 \quad \text{and} \quad i(0, \cdot) = i_0 \in L^2_+[0, a_+].$$

In the model (6.1), the population is decomposed into the class (S) of susceptible individuals and the class (I) of infected individuals. The number of individuals in the class (S) at time t is $S(t)$. The age of infection $a \in [0, a_+]$ is the time since the infection began (where a_+ denotes the maximum age of the epidemic), and $i(t, a)$ is the density of infected individuals with respect to the age infection. That is to say that for two

given age values $a_1, a_2 : 0 \leq a_1 < a_2 \leq a_\dagger$ the number of infected individuals with age of infection a between a_1 and a_2 is

$$\int_{a_1}^{a_2} i(t, a) \, da.$$

The function $\beta(a)$ is the force of infection at the age of infection a . We assume that there exists $a_b \in (0, a_\dagger)$ such that

$$(6.2) \quad \beta(a) = 0 \quad (a \in (0, a_b) \text{ a.e.}),$$

that is to say, we interpret infection age to an exposed period (infected, but not yet infectious) from $a = 0$ to $a = a_b$ and an infectious period from $a = a_b$ to $a = a_\dagger$. The quantity

$$\int_0^{a_\dagger} \beta(a) i(t, a) \, da$$

is the number of infectious individuals within the subpopulation (I). Finally, $\mu(a)$ is the exit (mortality and/or recovery) rate of infected individuals with an age of infection $a \geq 0$. As a consequence the quantity

$$\pi(a) := \exp\left(-\int_0^a \mu(\sigma) \, d\sigma\right)$$

is the probability for an individual to stay in the class (I) after a period of time $a \geq 0$.

System (6.1) can be seen as a semilinear Cauchy problem in an appropriate Banach space, in which the closed linear operator is not densely defined (see, for instance, Magal and Ruan [19, 20, 21]). In the above references, the existence and the uniqueness of solutions, for a given L^2 input function u , is proved by using a step-by-step procedure which preserves the positivity of solutions. The global-in-time existence and uniqueness follow since no finite time blowup occurs. Here, we aim to find a control function u such that the density of infected individuals in the age-infection model (6.1) becomes null in finite time. We suppose that $m = \mathbb{1}_{[0, a_0]}$ is the characteristic function of the interval $(0, a_0)$, with $a_0 \leq a_b$; this means that we are able to control infected individuals that are not yet infectious. The below proposition shows that by controlling the infected individuals (i.e., by using quarantine) before their infectiousness period we can eradicate the epidemic in finite time.

PROPOSITION 6.1. *Let assumption (6.2) be satisfied. Let $i_0 \in L^2[0, a_\dagger]$ with $i_0(a) \geq 0$ for almost every $a \in (0, a_\dagger)$ and let $S_0 \geq 0$. Then for every $\tau > a_\dagger - a_0$ there exists a control $u \in L^2([0, \tau]; L^2[0, a_\dagger])$ such that the controlled density of infected individuals i given by (6.1) satisfies*

$$i(\tau, a) = 0 \quad (a \in (0, a_\dagger) \text{ a.e.}).$$

Moreover the controlled density of infected individuals remains positive, that is to say,

$$i(t, a) \geq 0 \text{ a.e. } (t, a) \in Q_\tau.$$

By a slight variation of the calculations in section 2 we can check that the solution of (6.1) satisfies

$$(6.3) \quad S(t) = S_0 \exp\left(-\int_0^t \int_0^{a_\dagger} \beta(a) i(s, a) \, da \, ds\right)$$

and

$$(6.4) \quad i(t, a) := \begin{cases} \frac{\pi(a)}{\pi(a-t)} i_0(a-t) + c(t, a), & t \leq a, \\ \pi(a) \tilde{b}(t-a) + c(t, a), & a < t, \end{cases}$$

where

$$(6.5) \quad c(t, a) := \begin{cases} \int_{a-t}^a \frac{\pi(a)}{\pi(s)} m(s) u(s - (a-t), s) ds, & t \leq a, \\ \int_0^a \frac{\pi(a)}{\pi(s)} m(s) u(s + (t-a), s) ds, & a < t, \end{cases}$$

and $t \mapsto \tilde{b}(t)$ is the unique continuous function satisfying the Volterra integral equation

$$(6.6) \quad \tilde{b}(t) = S(t) \left\{ \int_0^{\min(t, a_\dagger)} \beta(a) [\pi(a) \tilde{b}(t-a) + c(t, a)] da + \int_{\min(t, a_\dagger)}^{a_\dagger} \beta(a) \left[\frac{\pi(a)}{\pi(a-t)} i_0(a-t) + c(t, a) \right] da \right\}.$$

Proof of Proposition 6.1. The remark suggesting the approach below is that, assuming S is a given function, the construction of the control in Theorem 4.2 and, more precisely, in Remark 4.3 adapts to the controllability of the system formed by the last two equations in (6.1). This motivates the construction of the control function below for system (6.1).

As previously, we focus on the case where $\tau \in]a_\dagger - a_0, a_\dagger[$.

Case $t - a < 0$. We first set

$$(6.7) \quad u(t, a) = \begin{cases} 0 & \text{if } t - a < \tau - a_\dagger, \\ \frac{i_0(a-t)}{\pi(a-t)} & \text{if } \tau - a_\dagger \leq t - a \leq 0, \\ \int_{a-t}^{\tau+(a-t)} \frac{m(x)}{\pi(x)} dx & \end{cases}$$

so that from (6.4) and (6.7) we have

$$(6.8) \quad i(t, a) = \begin{cases} \frac{\pi(a)}{\pi(a-t)} i_0(a-t) & \text{if } t - a < \tau - a_\dagger, \\ \frac{\pi(a)}{\pi(a-t)} i_0(a-t) \left[1 - \frac{\int_{a-t}^a \frac{m(x)}{\pi(x)} dx}{\int_{a-t}^{a+(\tau-t)} \frac{m(x)}{\pi(x)} dx} \right] & \text{if } \tau - a_\dagger \leq t - a \leq 0. \end{cases}$$

Formula (6.8) clearly implies that

$$(6.9) \quad i(\tau, a) = 0 \quad (a \in (\tau, a_\dagger) \text{ a.e.}).$$

Case $0 < t - a < a_b$. Using (6.8), we are able to compute $S(t)$ for every $t \in [0, a_b]$. Indeed, after an easy calculation using (6.8) it follows that for every $t \in [0, a_b]$,

$$(6.10) \quad \int_0^{a_\dagger} \beta(a) i(t, a) da = \int_{a_b}^{a_\dagger} \beta(a) i(t, a) da = \int_{t+a_\dagger-\tau}^{a_\dagger} \beta(a) \frac{\pi(a)}{\pi(a-t)} i_0(a-t) da.$$

In what follows, we set

$$(6.11) \quad f(t) := \int_{t+a_{\dagger}-\tau}^{a_{\dagger}} \beta(a) \frac{\pi(a)}{\pi(a-t)} i_0(a-t) da \quad (t \in [0, \tau]),$$

so that using (6.3), (6.10), and (6.11) we have

$$(6.12) \quad S(t) = S_0 e^{-\int_0^t f(s) ds} \quad (t \in [0, a_b]).$$

Then for every $t \in [0, a_b]$, using (6.2), (6.6), (6.10), and (6.12), we have

$$(6.13) \quad \tilde{b}(t) = S(t) \int_{a_b}^{a_{\dagger}} \beta(a) i(t, a) da = S_0 f(t) e^{-\int_0^t f(s) ds} \quad (t \in [0, a_b]).$$

Denoting by

$$(6.14) \quad \chi(t) := S_0 f(t) e^{-\int_0^t f(s) ds} \quad (t \in [0, \tau]),$$

it follows from (6.13) and (6.14) that we have $\tilde{b}(t) = \chi(t)$ for every $t \in [0, a_b]$. Then we set for almost every $(t, a) \in Q_{\tau}$ with $0 < t - a \leq a_b$

$$(6.15) \quad u(t, a) = \frac{\chi(t-a)}{\int_0^{\tau-(t-a)} \frac{m(x)}{\pi(x)} dx},$$

so that from (6.4) we have

$$(6.16) \quad i(t, a) = \pi(a) \chi(t-a) \left[1 - \frac{\int_0^a \frac{m(s)}{\pi(s)} ds}{\int_0^{\tau-(t-a)} \frac{m(s)}{\pi(s)} ds} \right] \quad \text{if } 0 < t - a \leq a_b.$$

It follows from (6.16) that we have

$$(6.17) \quad i(\tau, a) = 0 \quad (a \in (\tau - a_b, \tau) \text{ a.e.}).$$

Case $ja_b < t - a < (j-1)a_b$ for some integer $j \geq 1$. The above calculations suggest, as a possible control, driving the initial state to $u(t, a)$ where

$$(6.18) \quad u(t, a) = \begin{cases} 0 & \text{if } t - a < \tau - a_{\dagger}, \\ \frac{\frac{i_0(a-t)}{\pi(a-t)}}{\int_{a-t}^{\tau+(a-t)} \frac{m(x)}{\pi(x)} dx} & \text{if } \tau - a_{\dagger} \leq t - a \leq 0, \\ \frac{\chi(t-a)}{\int_0^{\tau-(t-a)} \frac{m(x)}{\pi(x)} dx} & \text{if } 0 < t - a \leq \tau, \end{cases}$$

where χ is defined by (6.14). With u defined by (6.18), it can be checked using a recursive argument as in the proof of Theorem 3.2 that we have

$$(6.19) \quad S(t) = S_0 e^{-\int_0^t f(s) ds} \quad (t \in [ja_b, (j+1)a_b]),$$

which implies that

$$(6.20) \quad \tilde{b}(t) = \chi(t) \quad (t \in [ja_b, (j+1)a_b]),$$

so that using (6.4), (6.18), and (6.20) we get

$$(6.21) \quad i(t, a) = \pi(a)\chi(t - a) \left[1 - \frac{\int_0^a \frac{m(s)}{\pi(s)} ds}{\int_0^{\tau-(t-a)} \frac{m(s)}{\pi(s)} ds} \right] \text{ if } ja_b < t - a \leq (j + 1)a_b,$$

and it follows from the above formula (6.21) that we have

$$(6.22) \quad i(\tau, a) = 0 \quad (a \in [\tau - (j + 1)a_b, \tau - ja_b] \text{ a.e.}).$$

Finally, with u defined by (6.18) we have

$$i(t, a) = \begin{cases} \frac{\pi(a)}{\pi(a-t)} i_0(a-t) & \text{if } t - a < \tau - a_{\dagger}, \\ \frac{\pi(a)}{\pi(a-t)} i_0(a-t) \left[1 - \frac{\int_{a-t}^a \frac{m(x)}{\pi(x)} dx}{\int_{a-t}^{a+(\tau-t)} \frac{m(x)}{\pi(x)} dx} \right] & \text{if } \tau - a_{\dagger} \leq t - a \leq 0, \\ \pi(a)\chi(t - a) \left[1 - \frac{\int_0^a \frac{m(s)}{\pi(s)} ds}{\int_0^{\tau-(t-a)} \frac{m(s)}{\pi(s)} ds} \right] & \text{if } 0 < t - a \leq \tau, \end{cases}$$

where we recall that we have defined for every $t \in [0, \tau]$

$$\chi(t) := S_0 f(t) e^{-\int_0^t f(s) ds},$$

so that the positivity of the controlled density i is clear. Moreover, it can be checked that we have $u \in L^2([0, \tau]; L^2[0, a_{\dagger}])$, thanks to Hardy’s inequality [12]. \square

Appendix A. Null controllability in $L^1[0, a_{\dagger}]$. In this appendix, we investigate the null controllability of system (1.1) when the state space X is chosen to be $L^1[0, a_{\dagger}]$ instead of $L^2[0, a_{\dagger}]$.

PROPOSITION A.1. *For $\tau > 0$ we denote by $(\mathbb{T}_{\tau})_{\tau > 0}$ and $(\Phi_{\tau})_{\tau > 0}$ the semigroup acting on $L^1[0, a_{\dagger}]$ and the control to state maps, defined on $L^1[0, \tau]$ with values in $L^1[0, a_{\dagger}]$, defined according to Remark 2.2. Then for every $\tau > a_{\dagger} - a_0$ we have that $\text{Ran } \mathbb{T}_{\tau} \subset \text{Ran } \Phi_{\tau}$, i.e., system (1.1), with state space $X = L^1[0, a_{\dagger}]$ and input space $U = L^1[0, \tau]$ is null controllable in any time $\tau > a_{\dagger} - a_0$.*

Proof. We can assume, without loss of generality, that $a_0 \leq a_b$. Let $\tau \in (a_{\dagger} - a_0, a_{\dagger})$. Given $p_0 \in X$, we look for a control $u \in L^1([0, \tau]; L^1[0, a_{\dagger}])$ such that the solution p of (1.1) satisfies $p(\tau, \cdot) = 0$. To this aim, we use the control constructed in Remark 3.3 (with $\tilde{p} = 0$) with a slight modification (see Figure 2 for an illustration of the modification). Let $\delta \in (a_{\dagger} - a_0, \tau)$. We set

$$(A.1) \quad u(t, a) = \begin{cases} 0 & \text{if } t - a < \delta - a_{\dagger}, \\ -\frac{\frac{p_0(a-t)}{\pi(a-t)}}{\int_{a-t}^{\tau+(a-t)} \frac{m(x)}{\pi(x)} dx} & \text{if } \delta - a_{\dagger} \leq t - a \leq 0, \\ -\frac{\eta(t-a)}{\int_0^{\tau-(t-a)} \frac{m(x)}{\pi(x)} dx} & \text{if } 0 < t - a \leq \delta, \\ 0 & \text{if } \delta < t - a \leq \tau, \end{cases}$$

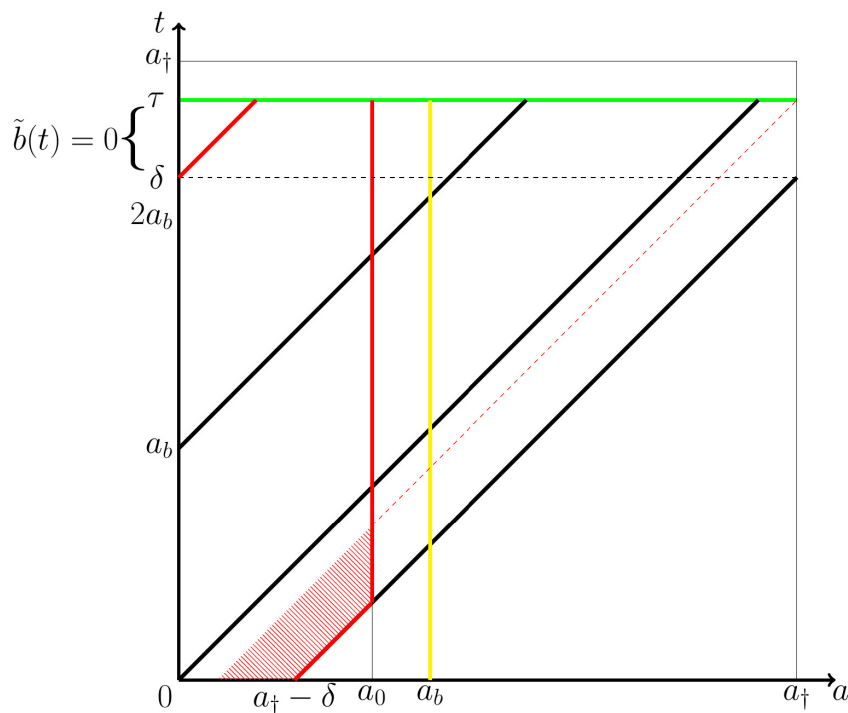


FIG. 2. In this figure, the control acts in the subset of $Q_\tau = [0, \tau] \times [0, a_\dagger]$ delimited by the line $a = a_0$ and the red curve. The hatched region corresponds to the slight modification of the control region (compared to the control pictured in Figure 1). Due to this modification, the value of $\tilde{b}(t) = \int_0^{a_\dagger} \beta(a)p(t, a) da$ is null for every $t \in [\delta, a_\dagger]$, and we can choose $u(t, a) = 0$ for $\delta \leq t - a \leq a_\dagger$ to avoid the singularity in the neighborhood of the point $(\tau, 0)$. (Color available online.)

where

$$\eta(s) = \int_{s+a_\dagger-\delta}^{a_\dagger} \beta(x)\pi(x) \frac{p_0(x-s)}{\pi(x-s)} dx \quad (s \in [0, \delta]).$$

With u defined by (A.1), it can be checked using a recursive argument as in the proof of Theorem 3.2 that we have

(A.2)

$$p(t, a) = \begin{cases} \frac{\pi(a)}{\pi(a-t)} p_0(a-t) & \text{if } t-a < \delta - a_\dagger, \\ \frac{\pi(a)}{\pi(a-t)} p_0(a-t) \left[1 - \frac{\int_{a-t}^a \frac{m(x)}{\pi(x)} dx}{\int_{a-t}^{a+(\tau-t)} \frac{m(x)}{\pi(x)} dx} \right] & \text{if } \delta - a_\dagger \leq t-a \leq 0, \\ \pi(a)\eta(t-a) \left[1 - \frac{\int_0^a \frac{m(s)}{\pi(s)} ds}{\int_0^{\tau-(t-a)} \frac{m(s)}{\pi(s)} ds} \right] & \text{if } 0 < t-a \leq \delta. \end{cases}$$

It follows from (A.2) that we have

$$(A.3) \quad p(\tau, a) = 0 \quad (a \in [a_\dagger - \delta, a_\dagger] \text{ a.e.}).$$

If $t \in [\delta, a_{\dagger}]$, it follows from (A.2) that we have

$$\tilde{b}(t) = \int_{a_b}^{a_{\dagger}} \beta(a)p(t, a)da = 0,$$

so that from (2.9) and (A.1) we get

$$(A.4) \quad p(t, a) = 0 \quad (\delta < t - a \leq \tau).$$

It follows from (A.2) and (A.4) that the controlled density of population p is given by

$$p(t, a) = \begin{cases} \frac{\pi(a)}{\pi(a-t)}p_0(a-t) & \text{if } t-a < \delta - a_{\dagger}, \\ \frac{\pi(a)}{\pi(a-t)}p_0(a-t) \left[1 - \frac{\int_{a-t}^a \frac{m(x)}{\pi(x)} dx}{\int_{a-t}^{a+(\tau-t)} \frac{m(x)}{\pi(x)} dx} \right] & \text{if } \delta - a_{\dagger} \leq t-a \leq 0, \\ \pi(a)\eta(t-a) \left[1 - \frac{\int_0^a \frac{m(s)}{\pi(s)} ds}{\int_0^{\tau-(t-a)} \frac{m(s)}{\pi(s)} ds} \right] & \text{if } 0 < t-a \leq \delta, \\ 0 & \text{if } \delta < t-a \leq \tau, \end{cases}$$

so that we have $p(\tau, \cdot) = 0$ and the control u defined by (A.1) is in $L^1([0, \tau]; L^1[0, a_{\dagger}])$. \square

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